NoBS Calculus

for Calculus 1, 2, and 3

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Contents

Author's Notes

0.1 NoBS

NoBS, short for ”no bull$#!%”, strives for succinct guides that use simple, smaller, relatable concepts to develop a full understanding of overarching concepts.

0.2 What NoBS Calculus covers

This guide succinctly and comprehensively covers most topics in an explanatory notes format for Calculus 1, 2, and 3 (or Differential, Integral, and Multivariable Calculus). Note we cover in a late transcendentals format. Exponentials and logarithms are taught in Chapter 7.

0.3 What NoBS Calculus does not cover

At this time, NoBS Calculus does not cover:

- Precalculus
- Hyperbolic functions
- Polar functions

0.4 What parts are relevant to me?

If you're studying calculus in a university, simply refer to the part corresponding to your course. For the most part, as long as your university offers the traditional three-course calculus sequence, each part should have the respective sections that your course teaches.

If you are taking AP Calculus AB, only the Calculus 1 part will be relevant to you. If you are taking AP Calculus BC, both Calculus 1 and Calculus 2 are relevant to you. Neither AP Calculus course covers Calculus 3: Multivariable Calculus.

0.5 A non-graphical approach

It is important to have a good image of everything we do here in your head. Not only is it a pain for us to add various graphs into this book, it is usually unnecessary. You should develop a mental image of all functions we graph here. If not, please consult your Algebra 1, Algebra

2, and Precalculus textbooks for a review on functions' graphs. Consider reading OpenStax's free online textbooks on Algebra and Precalculus. (Don't worry; nobody is judging you. We all sometimes need a refresher.) If you come across certain graphs that are too difficult to imagine (and they *will* pop up), please use Desmos or a capable graphing utility to gain a visual picture.

0.6 Dedication

To all those that helped me in life: this is for you.

A huge thank you to Garrett Gu for a major rewrite of our proof by induction section.

0.7 Sources

This guide borrows certain material from the following sources, which are indicated below and throughout the paper will be referenced by parentheses and their names:

- (Kallman) Robert R. Kallman, University of North Texas
- (Briggs Cochran) *Calculus, 2nd ed.*, Pearson Education
- (Sparling) http://math.pitt.edu
- (Wikipedia) Wikipedia, https://en.wikipedia.org

Prerequisites

Chapter 1

Precalculus review and proof by induction

If you have not yet learned Precalculus (Elementary Functions and Trigonometry), go away. Note that this is a review, not comprehensive notes, for those who have learned Precalculus. Be warned: it will be very brief and extremely succinct.

1.1 Functions

The ones you need to know

Please review the following functions' parent values and some attributes we will give after this list.

1. Constant
2. Linear
3. Quadratic
4. Cubic
5. Radical (square root)
6. Rational (inverse)
7. Absolute value
8. Logarithmic
9. Exponential

For each of these functions, you will need to remember the following attributes:

1. Domain and range
2. Horizontal and vertical asymptotes
3. If the function is odd, even, or neither
4. Continuity across $x \in \mathbb{R}$

1.2 Trigonometry

You will need to know all of the trigonometric functions, the unit circle, sinusoidal functions/their graphs, their behaviors, their characteristics, how to manipulate them, the trig identities, etc.

1.3 Proof by induction

with contributions by ***Garrett Gu***.

Because this is taught in some Precalculus classes but not all, and this is also taught as new in some other Calculus classes but not all, we will be explaining this section in detail.

Mathematical induction is sometimes useful when you have a set, and you need to prove that a statement holds for every element in the set. To be specific, you need a well-ordered set, which implies that there is some minimum value in the set.

Let's prove that the following statement (let's call it $S(n)$) holds for all natural numbers n ($n \in \mathbb{N}$). It just so happens that the set of natural numbers is a well-ordered set.

$$0 + 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

First, we must prove the **base case**. Let's see whether the statement holds true for the first n, which in this case is 0, because the smallest number in the set of natural numbers is 0. We will see if $S(0)$ is true.

$$0 = \frac{0(0+1)}{2}$$
$$0 = 0$$

So both sides are equal to each other and this statement holds true for now.

Now, let's prove the **inductive step**. We will assume that $S(n)$ is indeed true.

$$0 + 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Then, see if $S(n+1)$ holds true.

$$0 + 1 + 2 + 3 + \cdots + n + (n+1) = \frac{(n+1)(n+2)}{2}$$

Since we assumed that $0+1+2+3+\cdots+n = \frac{n(n+1)}{2}$ is true, we'll replace $0+1+2+3+\cdots+n$ with $\frac{(n+1)(n+2)}{2}$.

$$\frac{n(n+1)}{2} + (n+1) = \frac{(n+1)(n+2)}{2}$$

$$\frac{n^2+n}{2} + n + 1 = \frac{n^2+3n+2}{2}$$

$$\frac{n^2+n}{2} + \frac{2n}{2} + \frac{2}{2} = \frac{n^2+3n+2}{2}$$

$$\frac{n^2+n+2n+2}{2}=\frac{n^2+3n+2}{2}$$

$$\frac{n^2+3n+2}{2}=\frac{n^2+3n+2}{2}$$

We have shown that if $S(n)$ is true, $S(n+1)$ is also true. Since we have shown that $S(0)$ is true, we can plug $n=0$ into what we just proved to show that $S(1)$ is also true. We can then use that to show that $S(2)$ is also true, and so on. Mathematical induction works like a domino line. You knock over the domino at the beginning (the base case), and each domino knocks over the one after it, causing every domino to be knocked over.

Q.E.D.—we have just proved $0+1+2+3+\cdots+n=\frac{n(n+1)}{2}$.

(Also, you can also write Q.E.D. like this: $\Box$)

The hardest part of proof by induction is the problem solving part. Actually applying proof by induction to a problem is harder than learning proof by induction itself.

Part I

Calculus 1

Chapter 2

Limits

Before beginning limits, it is important to note that Calculus I is a conceptually-rooted course, as opposed to a memorization-based course. You cannot simply memorize the content taught in this course and expect to apply it to problems you see on your homework, quizzes, and tests.

Starting from this part of the notes, we will become more explanatory. This is not a review. It is straight-to-the-point explanation.

We begin our calculus journey with limits. It's *integral* (that was a calculus pun) you know limits extremely well, because your understanding of everything after limits depends on it.

2.1 Basics of limits

Definition of limits

A limit is the value of a function when it approaches a certain point on the graph but doesn't necessarily have to equal it. For example, in the function $f(x) = x$, as x approaches 0 (or gets closer to 0), $f(x)$ also approaches 0. But in the function $f(x) = \frac{x^2}{x}$, no value exists at $x = 0$ because it would be $\frac{0}{0}$, which is undefined. However, the function still approaches 0 even though it doesn't equal 0. We write this as:

$$\lim_{x \to 0} f(x) = 0$$

Note that in the trivial example $f(x) = x$, the following holds true: $f(c) = \lim_{x \to c}$, where c is a constant. (In set notation, $c \in \mathbb{R}$.)

How to determine limits

The easy way? Get your pencil out and follow the graph (in your head, if you don't do this on paper) from the left side to point c. (Point c is the point we're trying to approach and take the limit of.) Where does your pencil go? Even if $f(c)$ does not exist for a certain point, its one-sided limit will. Where your pencil ended up at point c is the limit. Now try following the graph from the right side to point c. Did it end up at the same place as the left-side approach? If so, that same place is the limit's result. If not, the two-sided limit (or a regular limit) does not exist for point c, but the one-sided limits exist.

One-sided limits

So where do you approach limits from? Regular limits are actually saying that when you approach a point from both the left and right, it will go to the same place. Sometimes, functions don't do this. $f(x) = \frac{1}{x}$ is an example. Take its approach to $x = 0$. From the right, it appears to approach $+\infty$ while from the left, it appears to approach $-\infty$.

We denote limits from the left as $\lim_{x\to c^-} f(x)$ while limits from the right are denoted as $\lim_{x\to c^+} f(x)$. (In this case, c is the name of the point that we are approaching.)

2.2 Strategies for finding limits

Canceling factors

$$f(x) = \frac{x^2 - 9}{x + 3}$$

Let's begin with this. If we try to evaluate the limit as we approach -3, we'll get indeterminate form.

$$\lim_{x\to -3} \frac{((-3)-3)((-3)+3)}{((-3)+3)} = \lim_{x\to -3} \frac{-6\times 0}{0} = \frac{0}{0}$$

This is an issue. Not only is anything divided by 0 "undefined," $\frac{0}{0}$ is a special "indeterminate form" and it won't get you anywhere. If I multiplied something by 0, the answer is 0. So if I divide 0 by 0, the answer is technically anything.

So what we will be doing instead is factoring this out so that the offending factor will go away and we can evaluate the limit properly.

$$\lim_{x\to -3} \frac{(x-3)\cancel{(x+3)}}{\cancel{(x+3)}}$$
$$= -3$$

Make no mistake: $f(-3)$ does not exist (because this strategy for limit evaluation does not apply to evaluating the value of regular functions) but we can compute the limit through this strategy.

Limit rules

Several rules can be applied to limits to make solving them easier. Basically, it involves either breaking them apart, or rarely, stitching them together. Just make sure you don't do anything mathematically illegal!

1. **Sum rule**:
$$\lim_{x\to c} \big(f(x) + g(x)\big) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$$
2. **Difference rule**:
$$\lim_{x\to c} \big(f(x) - g(x)\big) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$$
3. **Constant coefficient/constant multiple rule** (where λ is the constant):
$$\lim_{x\to c} (\lambda f(x)) = \lambda \lim_{x\to c} f(x)$$

4. **Product rule**:
$$\lim_{x\to c}\big(f(x)\cdot g(x)\big) = \lim_{x\to c} f(x)\cdot \lim_{x\to c} g(x)$$
5. **Quotient rule**:
$$\lim_{x\to c}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$$
6. **Exponent/power rule** (where n is the exponent/power):
$$\lim_{x\to c}(f(x)^n) = (\lim_{x\to c} f(x))^n$$
7. **Rational exponent/power rule** (where n is the numerator and m is the denominator of the power, $m \geq 0$, m is even, and n/m is reduced to lowest terms):
$$\lim_{x\to c}(f(x)^{\frac{m}{n}}) = (\lim_{x\to c} f(x))^{\frac{m}{n}}$$

Squeeze theorem

The squeeze theorem states that if, at point c, function f goes in between two other functions p and q, and $p(c) \to L$ and $q(c) \to L$, then $f(c) \to L$. (The symbol "$\to$" means "approaches".)

Example: Let $f(x) = x$, $p(x) = |2x|$, $q(x) = -|2x|$.
Since:
$$\lim_{x\to 0} p(x) = 0 = \lim_{x\to 0} q(x)$$
and $f(x)$ passes in between $p(x)$ and $q(x)$ at this point,
$$\lim_{x\to 0} f(x) = 0$$
This makes sense because if two functions bound a third function, and these two functions meet at a certain place, then the limit of the third in-between function would have to be that certain place. In the inequality $0 \leq x \leq 0$, x would have to be 0. It was "squeezed" to 0.

2.3 Infinite limits and limits at infinity

While these two things may seem like the same thing, the former refers to a certain point on the graph approaching either positive or negative infinity ($+\infty, -\infty$), while the latter refers to the function approaching a certain point at the ends of it (or end behavior).

Infinite limits

$$\lim_{x\to 0}\frac{1}{x^2} = +\infty$$
To evaluate whether a function approaches $\pm\infty$, let's evaluate the individual terms of the fraction. $\lim_{x\to 0}\frac{1}{x^2}\ \frac{\to 1}{\to 0}$. While $\frac{1}{0}$ does not exist, as x gets ever so closer to 0 from both the left side and the right side, the value of x^2 decreases. But 1 divided by this small number will yield a large number. (The terms small number and large number are defined as incredibly small and incredibly large; they aren't any particular value, but they are intuitively "felt" to be very small and very big.) This large number is basically approaching $+\infty$ at 0.

Limits at infinity

$$\lim_{x\to+\infty} \frac{5x^2+4}{x^2+4} = 5$$

Also known as "end behavior," limits at infinity look at where the graph tends to level off on. Basically, you are finding the horizontal asymptote.

Elementary functions-based approach:

- **BOBO**: Bigger on bottom, 0. If the degree of a polynomial in the numerator is less than the degree of the polynomial in the denominator, the horizontal asymptote is at $y = 0$.
- **BOTNO**: Bigger on top, no asymptote. This is technically incorrect, because there is a slant asymptote you can find based on evaluating the function based on synthetic division.
- **BOSCO**: Both same, coefficients. This means if both degrees are the same, then divide the numerator's term with the largest degree by the denominator's term with the largest degree. You should basically be dividing the two coefficients and the result will be the horizontal asymptote.

Degree division approach:

1. Find the degree of the polynomial. Divide each term, in both numerator and denominator, by x^n, where n is the degree.
$$\lim_{x\to+\infty} \frac{5+\frac{4}{x^2}}{1+\frac{4}{x^2}}$$
2. Evaluate. You should get some fractions to be approaching zero. ($\frac{4}{(+\infty)^2} = \frac{4}{+\infty} = 0$.)
3. $$\frac{5+0}{1+0} = \frac{5}{1} = 5.$$

2.4 Continuity

We say a function is continuous at a point c if three criteria are met: (Briggs Cochran Calculus)

1. $f(c)$ exists on the function f;
2. $\lim_{x\to c} f(x)$ exists; and
3. $\lim_{x\to c} f(x) = f(c)$, or the value of the limit is equivalent to the actual function's value.

If any of the three criteria are not met, then the function at point c cannot be continuous. That point will therefore be called a point of discontinuity (also called holes).

Let's see where we can find this point of discontinuity.

$$f(x) = \frac{(x+1)(x-1)}{(x-1)}$$

$f(x)$ is effectively the graph of $y = x + 1$ except at $x = 1$, the function will be undefined because we will reach indeterminate form $\frac{0}{0}$.

Let $c = 1$. This means $f(c) =$ undefined, and since it does not exist on the function, function f at point c does not exist, and therefore this function is not continuous at $x = 1$. There is a point of discontinuity (hole) at $c = 1$.

2.5 Epsilon-delta (ε, δ)

Epsilon-delta is a method of proving limits. It gives a precise definition of what a limit is.

So far, we've been using methods that we are *told* work, but we don't know whether they really work or not. In response to this, we must use epsilon-delta proofs to indeed prove that a limit exists.

The formal definition of (ε, δ) is that:

$$\lim_{x \to c} f(x) = L$$

means that for every $\varepsilon > 0$, there exists a $\delta > 0$, such that for every x, the expression $0 < |x - c| < \delta$ implies $|f(x) - L| < \epsilon$.

Let's break down what this definition means.

1. "for every $\varepsilon > 0$" means we must cover all ranges of ε; i.e., we have no control over ε and our proof must hold true for all epsilons.

2. "there exists a $\delta > 0$" means our proof will be finding the value of δ (which is related to the value of ε)

3. "the expression $0 < |x - c| < \delta$" is the point on which our proof should start. This means while **finding** our proof, we should **end** on this. Note that $|x - c| > 0$, meaning $x \neq c$ and $|x - c| < \delta$ means our c will be within the reasonable limit of δ.

4. "implies $|f(x) - L| < \varepsilon$" is where we end our proof, but when we find our proof, we must start finding it from here.

We shall now complete a few examples.

Epsilon-delta proof of a linear function

Let $f(x) = 3x - 4$. Prove that $\lim_{x \to 5} 3x - 4 = 11$.

First, let's find the proof.

1. First, we need to find δ, so we set up the ε: $|f(x) - L| < \varepsilon$ is $|(3x - 4) - 11| < \varepsilon$. As you may notice, we don't have 5 yet, which is where the limit approaches. We will find it soon.

2. $|3x - 15| < \varepsilon$

3. $|3(x - 5)| < \varepsilon$

4. $|3||x-5| < \varepsilon$

5. $|x-5| < \frac{\varepsilon}{3}$

6. We have "found" 5! And it matches c, so we divide the 3 over to the other side. Since $|x-5| < \delta$, we can therefore say that $\delta = \frac{\varepsilon}{3}$.

(Partially sourced from http://www.milefoot.com/math/calculus/limits/DeltaEpsilonProofs03.htm)

Chapter 3

Differentiation

Differentiation is the section of calculus dedicated to the study of derivatives. Since limits are part of the *core* definition of derivatives, they are also very important in the field of differential calculus.

Differentiation = derivatives.

3.1 Introduction to derivatives

A derivative is a function that represents the rate of change of another function.

For example, let's take $f(x) = x$. How much does the slope of this graph change? None. (What is the slope of the *change* of the function? 0.) This is because it continues going the same place without changing its slope. Therefore, its change in slope is constant and its derivative is 0.

But then let's take $g(x) = x^2$. This function does not continue going in the same direction at all points of the function. It actually starts getting steeper as $x > 0$. The change in the slope of the function turns out to be $g'(x) = 2x$.

How do we figure out the derivative, then?

Difference quotient

To find the slope of anything, we need the slope formula.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

How do we find the change in the slope of a function if the slope keeps on changing, like with x^2? We'll need to find the slope at a point, because the slope at a point won't change (and it's the closest we can get to finding any slope at all).

But wait, the slope at a point? The slope formula only allows us to find the slope using two different slopes. Besides, there are infinitely many slopes on a function whose slope keeps changing.

Well, it turns out that if we take a point x and a point infinitely close to it (let's call it $x + h$ where h is the distance between x and the infinitely-close point), we will get the rate of the function's change, or the derivative.

How do we get infinitely close? We're going to use limits and approach near 0. That way, we'll be as close to the point x as we can possibly be, while still allowing us to get a slope using the slope formula. We're going to plug these pertinent values into the slope formula, getting us the difference quotient. Taking the limit at 0 of this difference quotient gets us the derivative of the function.

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{(x+h)-x} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x)$$

Another version of the difference quotient looks like this:

$$\lim_{h\to a} \frac{f(x)-f(a)}{x-a} = f'(x)$$

It could be applied to problems such as this: (Kallman)

$$\lim_{x\to 1} \frac{\sqrt[3]{x}-1}{x-1} = \frac{d}{dx}(\sqrt[3]{x})|_{x=1} = \frac{1}{3}$$

Such a problem is immensely difficult to evaluate without connecting it to differentiation. Now, we can simply take the derivative of $\sqrt[3]{x}$ and plug in 1 to the derivative. The result would be equivalent to the original limit's result.

(By the way, the notation $\frac{d}{dx}(\sqrt[3]{x})|_{x=1}$ means "the derivative of $\sqrt[3]{x}$ evaluated at $x = 1$.")

Tangent line

The derivative is only the slope of the line, but if we actually want to let it touch the actual function, we have to add a few extra things (cough cough x and y value) to get the tangent line.

Remember in geometry (ugh I know) how a line tangent to the circle specifically only touched that circle at ONE point? This is kind of the same. Except, if we don't adjust it to actually touch the function, it might not. Like, the derivative of $f(x) = x^2 + 9$ is $f'(x) = 2x$. And let's say we're finding the tangent line at point $(2, 13)$. But $2x$ never touches $x^2 + 9$, which is a problem. So, we will need to use the **point-slope formula** (back from Algebra 1) to find out the tangent line's equation. To determine the slope from the derivative, plug in the point's x value into the derivative.

$$f'(2) = 2 \times 2 = m_{tan} = 4$$

Here is the point-slope formula:

$$y - y_1 = m(x - x_1)$$

Now, plug $(2, 13)$ into the point slope formula, as well as the slope.

$$y - 13 = 4(x - 2)$$

Simplify.

$$y = 4x - 8 + 13$$

$$y = 4x + 5$$

And that's the tangent line!

Normal line

Refer to the above example for tangent line. We just take the opposite reciprocal of the slope of the plugged-in point-slope equation.

$$y - 13 = \frac{-1}{4}(x - 2)$$

3.2 Differentiable or not?

- If f is not continuous at c, then f cannot be differentiated at c.
- If f has a sharp turn (corner) at c, then f cannot be differentiated at c.
- If f has a vertical tangent at c, then f cannot be differentiated at c.

(Briggs Cochran Calculus)

You can't always differentiate something at a certain point. It turns out that continuity is important for differentiation. Since you can't really take the slope of a "jump" discontinuity (i.e. when the function suddenly jumps somewhere else with no rhyme or reason), there is no derivative for this point.

Note that if f is continuous at a point, it doesn't necessarily mean that f can be differentiated at that point.

At sharp turns, the derivative of the left side (c approached from the left) will not be the derivative of the right side (c approached from the right), so therefore there is no derivative at c. Similar logic applies for vertical tangents.

3.3 Derivative notation

There are several ways to indicate you're taking the derivative of something.

1. Leibniz's notation: $\frac{d}{dx}(x^2)$ for the first derivative, $\frac{d^2}{dx^2}(x^2)$ for the second derivative, etc.
2. Lagrange's notation: $f'(x)$ for the first derivative, $f''(x)$ for the second derivative, etc.
3. Euler's notation: $Df = \frac{df}{dx}$ for the first derivative, $D^2f = \frac{d^2f}{dx^2}$ for the second derivative, etc.

Leibniz's notation is ideal for taking the derivative of an expression, but may be too complex to write out sometimes. Lagrange's notation is handy for simple first or second derivatives, but imagine taking 50 derivatives. 50 primes (the apostrophes) is simply uncountable. Euler's notation is used to find derivatives of functions in higher level analytic studies, where many orders of derivatives could be taken.

3.4 Rules for differentiating

It turns out that mathematicians found ~~lazier~~ more efficient ways, or patterns, to calculate derivatives. Notice many of these have the same idea and similar execution as limit rules, like the constant multiple/constant factor rule.

Constant rule

Since constants don't change their slopes, ever, they have 0 slope and therefore their derivatives will always be 0. In the below example, let $c \in \mathbb{R}$.

$$\frac{d}{dx}(c) = 0$$

Power rule

For any polynomial $p(x) = x^n$, its derivative is $p'(x) = nx^{n-1}$. This applies per term of a polynomial. So if my polynomial function were $f(x) = 3x^2 + 2x - 1$, its derivative would be $f'(x) = 6x + 2$. Remember that -1 "disappears" because its derivative, per the constant rule, is 0, and we don't need to add/subtract 0 to the function's derivative.

Constant multiple/constant factor rule

$$\frac{d}{dx}(\lambda f(x)) = \lambda f'(x)$$

Sum and difference rule

$$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$$

Product rule

$$\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + f'(x)g(x)$$

Quotient rule

$$\frac{d}{dx}(\frac{f(x)}{g(x)}) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

(Where $g(x) \neq 0$)

A good way to remember this is "low D-low, high D-low". Then divide and square the low.

Chain rule

The chain rule deals with taking the derivatives of compositions, like $(f \circ g)'(x)$.

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

You can use the chain rule to solve functions that seemingly cannot be compositions, but actually are. For example, if $h(x) = \sqrt{x+2}$, then $f(x) = \sqrt{x}$, $g(x) = x + 2$, and $h'(x) = f'(g(x)) \cdot g'(x)$ or $h'(x) = \frac{1}{2}x^{-\frac{1}{2}} \cdot (x + 2) \cdot 1$.

You can also "chain" multiple compositions together. Let $f(x) = \sin(\cos(x^7))$. What would $f'(x)$ be? Let $f(x) = (g \circ h \circ j)(x)$ where $g(x) = \sin(x)$, $h(x) = \cos(x)$, and $j(x) = x^7$. Therefore, $g'(x) = \cos(x)$, $h'(x) = -\sin(x)$, and $j'(x) = 7x^6$.

$$f'(x) = g'(h(x)) \cdot h'(x)$$
$$f'(x) = g'(\cos(x^7)) \cdot (\cos(x^7))'$$
$$h'(x) = h'(j(x)) \cdot j'(x)$$
$$(\cos(x^7))' = -\sin(x^7) \cdot 7x^6$$
$$f'(x) = \cos(\cos(x^7)) \cdot -\sin(x^7)6x^7$$
$$f'(x) = -6x^7 \sin(x^7) \cos(\cos(x^7))$$

3.5 Derivatives of trigonometric functions

- **Sine**:
$$\sin'(x) = \cos(x)$$
- **Cosine**:
$$\cos'(x) = -\sin(x)$$
- **Tangent**:
$$\tan'(x) = \sec^2(x)$$
- **Cosecant**:
$$\csc'(x) = -\csc(x)\cot(x)$$
- **Secant**:
$$\sec'(x) = \sec(x)\tan(x)$$
- **Cotangent**:
$$\cot'(x) = -\csc^2(x)$$

The proof for the derivative of the sine function is based on the difference quotient and then the squeeze theorem.

3.6 Derivatives in kinematics

As this guide is not made to teach physics, we will not be doing much in this section.

So here's what the central point is: given three functions, where function $s(t)$ denotes position at time t, function $v(t)$ denotes velocity at time t, and function $a(t)$ denotes acceleration at time t:

$$s'(t) = v(t)$$
$$v'(t) = a(t)$$
$$s''(t) = a(t)$$

3.7 Implicit differentiation

So far, this guide has been dealing with explicit functions, like $f(x) = x$. This is where the input value will define the output value. Seems straight forward, right?

Well, mathematicians ~~decided one day to become completely savage and mess up everything. You may have seen them already confuse you beginning in algebra class. After that, nothing made sense. Well, it's the exact same thing here! You're now going to dance around like monkeys—~~ needed a way to figure out how to solve *implicit* derivatives. This is where you ~~all of a sudden are supposed to know what two values will make each other equal to themselves~~ will be finding the derivatives relative to one of the variables.

$$x^2 + y^2 = 1$$

Because we can only take the derivative relative to one variable (for now), we will set y as a function of x. So, replace y with $y(x)$.

Then, taking the derivative of $y(x)$ requires the chain rule. $\frac{dy}{dx}$ exists because it is the change of y over the change of x. More information at this website (click here!). Can't click it? Go to `http://www.mathsisfun.com/calculus/derivatives-dy-dx.html`.

Returning to our original implicit function, let's take the derivatives of each term with this new method in mind.

$$(x^2)' + (y^2)' = (1)' \implies 2x + 2y\frac{dy}{dx} = 0$$

Now, kind of like how in basic algebra we kept the variables on one side and the constants on the other, let's put the terms containing $\frac{dy}{dx}$ on one side and the other terms on the other.

$$2y\frac{dy}{dx} = -2x$$

Divide away everything that isn't $\frac{dy}{dx}$ on the left to isolate $\frac{dy}{dx}$. Then simplify and you get your implicit derivative.

$$\frac{dy}{dx} = \frac{-2x}{2y} \implies -\frac{x}{y}$$

3.8 Related rates

Related rates is a concept *related* to implicit differentiation. Imagine two or more rates changing at the same time. The only way to calculate them is to implicitly differentiate these in respect to *time*.

Solve as you would an implicit equation, and then apply the word problems by using the following equivalencies:

- x = position of x-coordinate
- y = position of y-coordinate
- x' = velocity of x-coordinate (how the x-coordinate's position *changes*)
- y' = velocity of y-coordinate (how the y-coordinate's position *changes*)

Chapter 4

Applications of differentiation

4.1 The first and second derivatives

Before we begin talking about how to solve problems using derivatives, let's first establish what the importance of the first and second derivatives of a function are.

- The **first derivative** tells us where the extrema (maxima and minima) of a function are, as well as whether the slope of a region is decreasing or increasing.
- The **second derivative** tells us where the inflection points are (more on that later), as well as the concavity of a function.

Now, let's define some terms.

- **extrema** (singular: extremum): minima and maxima, where minima is the plural form of "minimum" and maxima is the plural form of "maximum." This is the same minimum and maximum as "$x^2 - 4x + 6$'s minimum is at (2,2)."
- **critical point**: an interior point c on the domain of a function where $f'(c) = 0$ or $f'(c)$ does not exist.
- **concavity**: whether a curve is opening up or down at a certain point. For example, the function x^2 is concave up (because the parabola opens upwards), while the function $-x^2$ is concave down (because the parabola opens downwards). When we start seeing polynomials such as $3x^4 - 2x^3 + 9x^2 - 7$, concavity becomes more difficult and less intuitive to solve. Note that linear lines do not have concavities, and neither do constant horizontal lines.
- **inflection points**: the point where a function's concavity shifts. This may not always exist in a function.

How do we link all of this together?

- When you set the **first derivative equal to 0**, you can find its **extrema**.
- When you analyze whether **intervals of the first derivative are positive or negative** between its extrema, you can determine that which intervals is the function's **slope increasing** (first derivative equals something positive) or **decreasing** (first derivative equals something negative).

- When you set the **second derivative equal to 0**, you can find its **inflection points**.
- When you analyze whether **intervals of the second derivative are positive or negative** between its inflection points, you can determine whether these intervals are **concave up** (second derivative equals something positive) or **concave down** (second derivative equals something negative).

4.2 Mean Value Theorems

Certain functions share the same derivative (e.g. $(x^2+2)' = (x^2)'$), and this is why we can manipulate this in our favor via the Mean Value Theorem!

Cauchy's Generalized Mean Value Theorem

On the open interval (a, b) of functions f and g, point c will exist so that this is true:

$$[f(b) - f(a)](Dg)(c) = [g(b) - g(a)](Df)(c)$$

or

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$$

We are able to prove l'Hôpital's rule (coming up later) using Cauchy's mean value theorem.

Lagrange's Mean Value Theorem

This is a more specific, applied version of Cauchy's mean value theorem.

$$f(b) - f(a) = (Df)(c) \cdot (b - a)$$

Lagrange's mean value theorem is actually just Cauchy's mean value theorem, but $g(x) = x$. Usually, this is more applicable, because we don't want to find some complicated polynomial function's equivalence of slope between points a and b.

This means if you find the slope of the line from point a to b, there will also be a point between a and b on this line that has the same tangent slope.

4.3 Defining extrema

Remember that the definition of extrema is that they have the least or greatest value out of all of the other points (in a certain interval, whether limited to a specific domain or throughout the entire function's domain). Theoretically, in order to prove an extrema is actually an extreme point, we must first determine whether the values of surrounding points are less or greater than the extrema's value.

Local extrema

Think of local extreme points that might not be the greatest maximum or least minimum of a function f.

Try graphing $f(x) = 3x^5 - 5x^3$ on Desmos (or your graphing calculator). While this guide does not provide a graph, nor do we promote it, we recommend you try visualizing it with a help of a graphing utility for this example. There is a local minimum at $(1, -2)$ and a local maximum at $(-1, 2)$. We will soon learn how to prove this.

Absolute extrema

Let a be all other points other than x, which are on the function f.

- If $f(x) > f(a)$, then x is the location of the absolute maximum of the function f.
- If $f(x) < f(a)$, then x is the location of the absolute minimum of the function f.

Extreme value theorem

On a closed interval [a,b], a function will have an absolute maximum and minimum value either somewhere between a and b or at a or b.

(Note: (a,b) would be an open interval (exclusive of a, b but inclusive of everything between a, b), while [a,b] would be a closed interval (includes a, b, and everything in between a,b.))

4.4 Finding local extrema

First derivative test

Now, let l and r be points left and right of point x and close enough to x such that they are within the same "curve."

If $f'(l) > 0$ and $f'(r) < 0$, then $(x, f(x))$ is a local maximum. (In other words, if the first derivative goes from above $y = 0$ to below it, there is a maximum where it crosses $y = 0$.)

If $f'(l) < 0$ and $f'(r) > 0$, then $(x, f(x))$ is a local minimum. (If the first derivative goes from below $y = 0$ to above it, there is a minimum where it crosses $y = 0$.)

Second derivative test

Let c be a point on the function f.

If $(D^2 f)(c) < 0$, then f has a strict local maximum at c.

If $(D^2 f)(c) > 0$, then f has a strict local minimum at c.

(Yes, that means if the second derivative at c is negative, there is a local maximum, and if c is positive, there is a local minimum. This is because at a local maximum, the graph just keeps changing to go downwards. Similarly, the graph will keep trying to change and go upwards at a local minimum.)

If $(D^2 f)(c) = 0$, then it's neither. (If $(D^2 f)(c) = 0$, then c could be just a regular inflection point or a saddle point.)

4.5 Finding extrema by differentiation

Since the slope at extrema is zero and derivatives are the slopes of a function, the **zeroes**/roots of a function's derivative *are where the extrema of the function are located*. This means we can find any function's extrema by taking its first derivative (taking a derivative one time) and setting it equal to zero. This will also help us find the most *optimized* values of a certain function, since extrema represent the best/worst case scenario of many possibilities.

4.6 Finding inflection points

To find inflection points, take the second derivative at a point.

If there is a certain interval where the second derivative is positive, then that interval is considered **concave up**.

If there is a certain interval where the second derivative is negative, then that interval is considered **concave down**.

There exists an inflection point if and only if

1. the second derivative is equal to 0, and
2. the function changes concavity there.

4.7 Introduction to optimization

One of the reasons why we learn differentiation of functions is because we want to calculate different scenarios based on certain parameters. This is important in the industry because we want to make sure we're getting the most out of what we've been given.

Find the relation between the different variables you're given. Then, in terms of just one variable, solve for a resulting answer. You want to optimize this resulting answer.

A simple example of optimization

John has been given 40 ft of fencing. He wants to fence the most area possible with this amount of fencing, in a rectangle. What will the width of this fenced area be?

Solution: We will be solving for the maximum area. The formula for a rectangle's area is length multiplied by width. So $A(w) = lw$. We need to determine length in terms of width, because that's what we're solving for. Furthermore, the maximum length and width can be is 40. So $2(l + w) = 40$. This means $l + w = 20$. Rearrange this in terms of l so we can plug the other side of the equation into the area equation. $l = 20 - w$. Therefore, $A(w) = w(20 - w)$. Now, we take the derivative of this function and set it equal to 0. The result will be the best width optimal for the most area.

$$A'(w) = (w)'(20 - w) + w(20 - w)'$$

$$A'(w) = 20 - w + w(-1)$$

$$A'(w) = 20 - 2w$$

Now, set $A'(w) = 0$.

$$0 = 20 - 2w$$

$$2w = 20$$

$$w = 10$$

The best width, then, is 10 ft. This will maximize area to be 100 ft^2. Any increase in width will cause the length to decrease, and consequently, the area will be less than 100 ft. Also, any decrease in width will cause the length to increase, and the area will be less than 100 ft.

4.8 L'Hôpital's rule

If a limit goes to indeterminate form ($\frac{0}{0}$ or $\frac{\infty}{\infty}$), you can take the derivative of the numerator and the derivative of the denominator. The limit of the differentiated fraction is the same as the limit of the original fraction.

This stems from Cauchy's generalized mean value theorem. Cauchy's states that:

$$[f(b) - f(a)](Dg)(c) = [g(b) - g(a)](Df)(c).$$

If we rearrange this, assuming $(Dg)(c) \neq 0$ and $g(b) - g(a) \neq 0$, then we get:

$$\frac{(Df)(c)}{(Dg)(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Now replace b with x, and assume $f(a)$ and $g(a)$ "vanish" (Sparling, math.pitt.edu). The right side, being the difference quotient, can be rearranged to become this with limits:

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = \lim_{x \to a} \frac{f(x)}{g(x)}$$

VERY IMPORTANT: <u>Do not confuse l'Hôpital's rule with the quotient rule.</u> You do not do low D-high minus high <u>D-low for l'Hôpital's and you do not use l'Hôpital's for</u> quotient rule.

By the way, l'Hôpital is pronounced "low-pee-tahl," not "le hospital" or "loopy towels."

Example of L'Hôpital's rule

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \frac{0}{0}$$

While we can easily cancel factors on top and bottom after we factor the numerator, let's try differentiating the numerator and denominator of the fraction (once on top, once on bottom, not using quotient rule!) and then evaluate the limit. That's the way to do it with l'Hôpital's rule.

$$\lim_{x \to 3} \frac{(x^2 - 9)'}{(x - 3)'} = \lim_{x \to 3} \frac{2x}{1} = \lim_{x \to 3} \frac{2(3)}{1} = 6$$

Now, let's try this next example:

$$\lim_{x \to \frac{\pi}{2}} \frac{1}{\cos(x)} = \frac{1}{0}$$

Initially, plugging in the limit will get you $\frac{1}{0}$. Since it was not $\frac{0}{0}$ or $\frac{\infty}{\infty}$, **you cannot use l'Hôpital's rule on this problem**! You will have to evaluate this limit the regular way. By the way, it doesn't exist, because the limit from the left does not equal the limit from the right.

Chapter 5

Integration

5.1 Antiderivatives

Antiderivatives: where to begin? Basically, you're finding the thing whose derivative is the function you have.

Let there be a function $f(x)$. Let $F'(x) = f(x)$. Then what is $F(x)$? This is the definition of an antiderivative.

Constant of integration

So it turns out that $\frac{d}{dx}(x^2) = 2x$. Well, same with $\frac{d}{dx}(x^2 + 1) = 2x$.

This means that there are actually infinite antiderivatives, because any constant could be tacked on to the antiderivative and you'd still end up with the same derivative. We don't know what this constant might be, so we tack on a big 'ol C at the end of all antiderivatives.

Finding antiderivatives using derivatives and logic

Let's say I have a function $f(x) = 2x$. What function's derivative is $2x$?

The answer is x^2. So $\frac{d}{dx}(x^2) = 2x$.

Alternatively, we could write $\int 2x \,\mathrm{d}x = x^2 + C$. That big 'ol C would be the constant of integration.

Let's try a trigonometric function. What trigonometric function's derivative is $\cos(\theta)$?

The answer is $\sin(\theta)$! This is because $\frac{d}{dx}(\sin(\theta)) = \cos(\theta)$. Similarly, this means $\int \cos(\theta) \,\mathrm{d}x = \sin(\theta)$.

5.2 Integration notation

Given this:

$$\int_a^b f(x) \,\mathrm{d}x$$

- $\int$ - this is the integral sign. It's a fancy olde-style cursive "S".

- $\int\limits_a^b$ - a and b are the limits of integration. These will only appear on definite integrals. On indefinite integrals, these are not present.
- $f(x)$ - this is the integrand, or the function being integrated.
- $\mathrm{d}x$ - the variable after the "d" determines what we are integrating in respect to. Other variables will be treated as constants. For definite integrals, you can imagine each of the numbers between the limits of integration being multiplied by the integrand. Don't worry about what this means for now though.

5.3 Rules for finding antiderivatives

Constant multiple/constant factor rule

This comes from the derivative version of the constant multiple rule.

$$\int \lambda f(x)\,\mathrm{d}x = \lambda \int f(x)\,\mathrm{d}x$$

Sum rule

Just like the derivative's sum rule, the sum rule applies to antiderivatives too.

$$\int f(x) + g(x)\,\mathrm{d}x = \int f(x)\,\mathrm{d}x + \int g(x)\,\mathrm{d}x.$$

Power rule

So remember the power rule for finding derivatives? It was $\frac{d}{dx}(x^n) = nx^{n-1}$.

The power rule for antiderivatives is:

$$\int x^n\,\mathrm{d}x = \frac{x^{n+1}}{n+1} + C.$$

Where is the chain rule?

Eh eh, not so fast. You can't just flip the derivative chain rule and call it the integral chain rule. We'll need to do something special later called *integration by substitution*, but for now, there really isn't a "chain rule" for integrals.

5.4 Finding the area under the curve

We know that the *slope* of a curve is represented by derivatives. Therefore, the opposite of derivatives, integrals, should be the *area* under the curve.

Why do we find the area under the curve?

Let's take the function $f(x) = 2x$, a linear function with slope of two. We are now going to find the area under the curve from 0 to x.

Let $x = 1$. From 0 to 1, we have a right triangle with base 1 and height 2. Remember that since $y = 2x$ that x represents the base and y represents the height. So, the area is $\frac{1}{2}bh = \frac{1}{2} \times 1 \times 2 = 1$.

Now, let $x = 2$. From 0 to 2, we have a right triangle with base 2 and height 4. So, the area is $\frac{1}{2}bh = \frac{1}{2} \times 2 \times 4 = 4$.

Let us evaluate one more time at $x = 3$. We can see that from 0 to 3, there will be a right triangle with base 3 and height 6. So, the area is $\frac{1}{2}bh = \frac{1}{2} \times 3 \times 6 = 9$.

If we let the function $F(x)$ be defined as the area from 0 to x, then we get from polynomial regression that $F(x) = x^2$.

$$x = 1, f(1) = 1^2 = 1$$

$$x = 2, f(2) = 2^2 = 4$$

$$x = 3, f(3) = 3^3 = 9$$

It is also important to note that $F'(x)$, the derivative of F, is $f(x) = 2x$. This means the area under the curve of $f(x)$ is $F(x)$, and therefore $F(x)$ is the antiderivative of $f(x)$.

You can therefore find the antiderivative by finding the area under the curve.

Riemann sums

While we have just accurately found the antiderivative of $f(x) = 2x$ (which is $F(x) = x^2$), sometimes, we aren't able to so smoothly integrate functions. Because we knew that the area under the curve formed by a linear function with y-intercept of zero is a triangle. However, there is no immediately obvious analytic way to calculate the area of an actually curvy function. This is where Riemann sums come in. (Pronounced: Ree-MAHN)

Riemann sums are approximations of area. Imagine you are taking the area under some curve defined by $f(x)$ on the interval $[a, b]$. We are going to now find a rectangle that spans the entire interval and is close to the area of the curve. Let's make its width $b - a$. At this point, the height, depending on which technique we've decided to use, will be determined by either (1) the leftmost point of the rectangle (which is $f(a)$), (2) the rightmost point of the rectangle (which is $f(b)$), or (3) the middle point of the rectangle ($a + \frac{b-a}{2}$). Remember that the leftmost point of the rectangle is also the least value of the interval.

Ok, so this isn't a very accurate way of determining the area under the curve. Now, let's split into two rectangles, one whose width goes from a to c. Let's let c be the middle point between a and b. Then, the other one will have a width that goes from c to b. This time, each rectangle's height will be determined by its own leftmost point, rightmost point, or midpoint. Then add the two areas of the rectangle together. Because the slope is likely at different heights at different parts of this interval, this approximation is more accurate than the one rectangle version.

To make this approximation more accurate, we will need to make more rectangles, which in turn reduces the width. Eventually the number of rectangles approach infinity (∞) and the widths of the rectangles approach 0. But the rectangles can then have their own height. When we add these rectangles' areas together, it more accurately corresponds with the true area of the curve than the original version we had. This *convergence* of values we will get as the widths approach 0 is the true area under the curve. (Note that the value will get more and more specific and will eventually *converge* to a value. Think of it as a limit approaching a value. As Riemann sums involve discrete widths and a finite amount of rectangles, they can not actually reach the area of the curve, but they will approach it.)

We will eventually learn a more accurate and simpler way to find areas under the curve, but Riemann sums will work for ANY and ALL curves that we can visualize on a graph. In fact, as long as a function is continuous, it is **Riemann integrable**. This is proven using epsilon-delta. However, it won't work with functions that are crazy, such as $f(x) = 1, x \in \mathbb{Q}; f(x) = 0, x \notin \mathbb{Q}$. ($\mathbb{Q}$ means all rational numbers.) Since there are infinitely many irrational numbers between any two rational numbers, this graph is certainly not continuous. It will jump between 1 and 0 all the time.

Here's a more confusing but condensed explanation. It might help you, but beware if you are already kind of confused. ~~Riemann sums basically are divided rectangles, usually uniformly divided but not necessarily so, that altogether span the interval that you are integrating. Each rectangle's height depends on the function's value there. The sum of the areas of the rectangles approximate the area under the curve. So, if the rectangles are of differing heights, they will be able to somewhat closely represent the area of the curve near that point. Alas, they are rectangles and not curves, so they certainly won't be 100% accurate. However, if we split the interval into more and more rectangles that each have their own corresponding height to the value of the curve at that point, then the area of the curve will get more accurate.~~

If you are still confused, it's probably better if you searched up a graphic, or better, a video, of what Riemann sums are. Honestly, Riemann sums are just much easier to visualize than explain with solely words.

5.5 Rules of definite integrals

Definite integrals basically mean finding the area under the curve. Ok, we've already went over Riemann sums, which is the computational way to find curves. That will *always* work. However, there are more efficient analytical methods to calculate definite integrals. Let's first begin with the basic rules of definite integrals.

1. By definition, if $b < a$, then:

$$\int\limits_a^b f(x)\,\mathrm{d}x = -\int\limits_b^a f(x)\,\mathrm{d}x$$

(Notice that the positions of a and b on the integral have been swapped between the two sides of the equation.)

2. **Sum rule**:

$$\int\limits_a^b (f+g)(x)\,\mathrm{d}x = \int\limits_a^b f(x) + g(x)\,\mathrm{d}x = \int\limits_a^b f(x)\,\mathrm{d}x + \int\limits_a^b g(x)\,\mathrm{d}x$$

3. **Constant multiple rule**:

$$\int\limits_a^b \lambda f(x)\,\mathrm{d}x = \lambda \int\limits_a^b f(x)\,\mathrm{d}x$$

(where λ is a constant)

4. $a \le b$ and $f(x) \ge 0$ for all $x \in [a,b] \implies$

$$\int_a^b f(x)\,\mathrm{d}x \ge 0$$

5. $a \le b$ and $f(x) \ge g(x)$ for all $x \in [a,b] \implies$

$$\int_a^b f(x)\,\mathrm{d}x \ge \int_a^b g(x)\,\mathrm{d}x$$

6. **Absolute value**: $a \le b \implies$

$$\left|\int_a^b f(x)\,\mathrm{d}x\right| \le \int_a^b |f(x)|\,\mathrm{d}x$$

7. **Subinterval additive rule**:

$$\int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x$$

8. **Constant rule**:

$$\int_a^b \lambda\,\mathrm{d}x = \lambda(b-a)$$

9. $|f| \le C \implies$

$$\left|\int_a^b f(x)\,\mathrm{d}x\right| \le C|b-a|$$

10. $$\int_0^1 x^n\,\mathrm{d}x = \frac{1}{n+1}$$

It is also possible to evaluate integrals using sums of limits.

5.6 Fundamental Theorem of Calculus

Warning: Both of the following theorems are ~~YUUUUUUUUGE!!!~~ very important to future explorations of the integral. Be sure you understand the Fundamental Theorem of Calculus inside and out.

What we observed when finding the area under the function $f(x) = 2x$ was that we would take the area from 0 to whatever x was. This lets us define the function:

$$A(x) = \int_a^x f(t)\,\mathrm{d}t$$

0 was the value of a in our case. This function treats t as a dummy variable, and indeed, it does not matter what t is. We are just trying to find the area under $f(t)$ by using x, and it solely depends on x, not t. This is because when we evaluate a definite integral, we plug in the limits of integration into the integrated integrand, and therefore $f(t)$ will be "replaced"; t is solely used to be integrated, and then it will be replaced.

First part of the Fundamental Theorem of Calculus

If f is continuous on $[a, b]$, then

$$A(x) = \int_a^x f(t)\,\mathrm{d}t$$

(for $a \leq x \leq b$)

$$A'(x) = \frac{d}{dx}\int_a^x f(t)\,\mathrm{d}t = f(x)$$

Also, the antiderivative of $f(x)$ exists; the first part of the fundamental theorem states that every continuous function f has an antiderivative F.

This is shown by the example in the subsection *Why do we find the area under the curve?* (in section 5.4 (click on 5.4 to get there), on page 29).

Note that if you had a function $g(x) = \int_a^{\Phi(x)} f(t)\,\mathrm{d}t$, then $g'(x) = f(\Phi(x))\Phi'(x)$. Therefore, it is important to apply the chain rule to the upper limit of integration when taking the integral's derivative. Do not apply the chain rule to the integrand.

Second part of the Fundamental Theorem of Calculus

If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

Hard to believe?

Let's try the example $\int_0^1 2x\,\mathrm{d}x$. Per our original example where we proved that the area under the curve from 0 to x is the integral of that function (in section 5.4, on page 29), the area from 0 to 1 should be 1. Well:

$$\int_0^1 2x\,\mathrm{d}x = 1.$$

$$\int_0^1 2x\,\mathrm{d}x = x^2\big|_0^1 = (1)^2 - (0)^2.$$

And so it holds. Try again, perhaps?

$$\int_0^2 2x\,\mathrm{d}x = \frac{1}{2} \times 2 \times 4 = 4.$$

$$\int_0^1 2x\,\mathrm{d}x = x^2\big|_0^2 = (2)^2 - (0)^2 = 4.$$

It would be fair to say that since the area from 0 to 1 is 1 and the area from 0 to 2 is 4 that the area from 1 to 2 is 3. Let's prove that, just to show you we're not proving only the first part of the theorem.

$$\int_1^2 2x\,\mathrm{d}x = x^2\big|_1^2 = (2)^2 - (1)^2 = 4 - 1 = 3.$$

5.7 Even and odd integration

For even functions (that is, functions with even symmetry), if you are integrating from $-a$ to a, then because the area on the left side of the y-axis is the same as the area of the right side of the y-axis, you can simply integrate from 0 to a and multiply that value by 2.

$$\int_{-a}^{a} f_{even}(x)\,\mathrm{d}x = 2\int_0^a f_{even}(x)\,\mathrm{d}x$$

For odd functions, if you are integrating from $-a$ to a, remember that any area below the x-axis is negative. So if both sides are symmetrical but one side is negative and the other is positive, they will cancel out and this integral will equal 0.

$$\int_{-a}^{a} f_{odd}(x)\,\mathrm{d}x = 0$$

5.8 Mean value theorem for integrals

Imagine a plane's altitude is at first 5,000 ft, goes up to 30,000 ft, then goes down to 15,000 ft. The function of the plane's altitude will have an average altitude.

All functions will have an average value, and this can be found using the mean value theorem for integrals.

$$f(c) = \bar{f} = \frac{1}{b-a}\int\limits_a^b f(t)\,\mathrm{d}t$$

At some point c will be the average value of the function. Also, $f(c)$ is equal to $\bar{f}$ (this means average value of function f). It can be found using the right-hand side of that equation, the side with the fraction and integrals.

5.9 Integration by substitution

Integration by substitution is basically the functional equivalent of the chain rule from differential calculus, but for integrals. It is also known as u-substitution. Just like how the chain rule made use of functions' *compositions* (e.g. things like $f(g(x))$, etc.), integration by substitution will also make use of $f(g(x))$, except the steps are different.

1. Start by determining your $g(x)$. Assign this to the variable u.
2. Take the derivative of $g(x)$, multiply it by dx, and set this equal to du.
3. Plug in du and u into the integral and integrate everything.
4. Replace u with what $g(x)$ originally was.
5. Finish integrating, and if applicable, evaluating the definite integral.

The substitution rule for indefinite integrals is

$$\int f(g(x))g'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u.$$

The substitution rule for definite integrals is

$$\int\limits_a^b f(g(x))g'(x)\,\mathrm{d}x = \int\limits_{g(a)}^{g(b)} f(u)\,\mathrm{d}u.$$

The substitution rule for definite integrals can be found either by substituting for an indefinite integral, then evaluating once you're done integrating, or you can use the change of variables rule, which is covered below.

An example of integration by substitution

What is:

$$\int \frac{x}{\sqrt{x+1}}\,\mathrm{d}x$$

(Briggs Cochran) Solution:

$$u = \Phi(x) = x + 1$$

$$\mathrm{d}u = (\Phi(x))' = (\Phi'(x)x') = 1\,\mathrm{d}x = \mathrm{d}x$$

$$\int \frac{x}{\sqrt{u}}\,\mathrm{d}u = \int \frac{u-1}{\sqrt{u}}\,\mathrm{d}u$$
$$= \int u^{1/2} - u^{-1/2}\,\mathrm{d}u = \frac{2}{3}u^{3/2} - 2u^{1/2}$$
$$= \frac{2}{3}(x+1)^{3/2} - 2(x+1)^{1/2}$$
$$= \frac{2}{3}(x+1)^{1/2}(x-2) + C$$

It also doesn't matter what you substitute. u could've also been $= \sqrt{x+1}$. It would be less convenient, but you'd get $\int \frac{u^2-1}{\sqrt{u}}\,\mathrm{d}u$ instead of $\int \frac{u-1}{\sqrt{u}}\,\mathrm{d}u$ (Briggs Cochran). This is because the u is different, and so the original integral must now "adapt" to the new u.

Do note that sometimes, $\mathrm{d}u = u'\,\mathrm{d}x$. It is important to isolate $\mathrm{d}x = \frac{\mathrm{d}u}{u'}$ so that it properly affects the integral. This is because, as you will find out later, *chain rule is part of integration by substitution*.

Change of variables

(The following content was borrowed from Wikipedia's article on integration by substitution.) Consider the integral

$$\int_0^2 x\cos\left(x^2+1\right)dx$$

If we apply the formula from right to left and make the substitution $u = \Phi(x) = x^2 + 1$, we obtain $\mathrm{d}u = 2x\,\mathrm{d}x \implies x\,\mathrm{d}x = \frac{1}{2}\,\mathrm{d}u$

$$\int\limits_{x=0}^{x=2} x\cos\left(x^2+1\right)\mathrm{d}x = \frac{1}{2}\int\limits_{u=1}^{u=5} \cos(u)\,\mathrm{d}u = \frac{1}{2}(\sin(5) - \sin(1)).$$

It is important to note that since the lower limit $x = 0$ was replaced with $u = 0^2 + 1 = 1$, and the upper limit $x = 2$ replaced with $u = 2^2 + 1 = 5$, a transformation back into terms of x was unnecessary.

(The preceding content was borrowed from its respective Wikipedia article.)

Chapter 6

Applications of integration

6.1 Integration in kinematics

- The derivative of position is velocity,
- The derivative of velocity is acceleration,

and as such,

- The integral of velocity is position,
- The integral of acceleration is velocity.

The issue with integration is that there are infinitely many antiderivatives, hence the constant of integration (C). However, physics requires that C to be a finite number, reducing possibilities of the integrated function from infinite to just one.

For finding the position function based off of the velocity function AND $s(0)$:

$$s(t) = s(0) + \int_0^t v(x)dx.$$

You can think of $s(0)$ as the y-intercept of $s(t)$ (which it is).

Similarly, for finding the velocity function based off of the acceleration function, you can use:

$$v(t) = v(0) + \int_0^t a(x)dx.$$

6.2 Regions between curves

Let's say we have an upper and lower curve on a graph. The upper curve's area under it will have a region that overlaps the lower curve's area under it. But there is also a region where only the upper curve has area. This unique region is actually between the upper and lower curve, because what's above the lower curve is part of the upper curve until the curve itself. Taking the difference of the lower curve from the upper curve (so $upper - lower$) results in the area of the region between curves.

A simple example of regions between curves

For simplicity's sake, let's make the upper curve $f(x) = 3$ and the lower curve $g(x) = 2$. What is the area of the region between the two curves from 0 to 4?

$$\int\limits_0^4 f(x) - g(x) = [3x - 2x]\Big|_0^4 = 4$$

A slightly more complicated example of regions between curves

What is the area of the region between the curves $f(x) = x^2 - 4$ and $g(x) = 4x$? First, we need to find out where the two curves intersect so we can determine where there is a closed region between the two curves. Set the two equal to each other. You get $x^2 - 4 = 4x \implies x^2 - 4x - 4 = 0 \implies (x-2)^2 = 0$. It looks like we will be integrating from -2 to 2.

Okay, now let's do the actual integral. Because $f(x) = x^2 - 4$ is on the top, we need to put this first, then put the subtraction symbol, and finally $g(x) = 4x$.

$$\int\limits_{-} 2^2x^2 - 4 - (4x) = [\frac{x^3}{3} - 4x - 2x^2]\Big|_{-2}^2$$

6.3 Volume by slicing

Volume by slicing is equivalent to a noncircular form of volume by rotation. The formula looks basically like this:

$$\int\limits_a^b lh \, \mathrm{d}x$$

Where l is length, h is height, and $\mathrm{d}x$ is the height/third dimension of the shape.

Volume by rotation, disk method

Imagine the area under the curve is an infinitely thin sliver that sits on the x-axis. Now, imagine that the x-axis is a skewer (like for kebabs) and you rotate this entire area about the x-axis skewer. This is the principal idea behind volume by rotation. The solid generated from the rotation of the area about the x-axis is called a **solid of revolution**.

An example of the disk method: cone

If you did this with the function $f(x) = 2x$ from 0 to 5, then if you rotated the area under the curve about the x-axis, there would be a cone. The pointy end of the cone would be at $x = 0$, and the base would be at $x = 10$.

Let's try finding the area of this cone by using integrals.

First, we know that:

- dx represents the length of the infinitely skinny sliver (whose height goes from the MIDDLE of the cone to the slant edge). If you were to rotate this about the x-axis, you get a cylinder with basically 0 height. This cylinder's base is a circle and it has the radius r, which is the finite height, or 2x.

- 2x represents the finite height of the sliver from the MIDDLE of the cone to the slant edge. If you think about it, this finite height is also the radius of the circle located at point "x" of the graph. Since this is the radius, it needs to be squared.

- It wouldn't be a circle without π. So we add that too.

$$V = \pi \times r^2 \times h = \int_0^5 \pi 4x^2 \, \mathrm{d}x$$

$$V = \frac{4}{3}\pi x^3\Big|_0^5 = \frac{4}{3}\pi(5)^3 - \frac{4}{3}\pi(0)^3 = \frac{125 \times 4 \times \pi}{3} = \frac{1}{3} \times 500 \times \pi = \frac{500\pi}{3}.$$

Let's prove this by solving for this cone using the well-known geometric formula.

$$V = \frac{1}{3}\pi r^2 h\Big|_{r=10,h=5} = \frac{1}{3} \times \pi \times 100 \times 5 = \frac{500\pi}{3}.\square$$

More complex examples of volume by rotation

A cone is quite easily to geometrically determine, so it may not be necessary to determine its volume using rotation. What if the curve weren't as straightforward as a linear function? (Pun not intended.)

This is why we cannot always depend on geometric methods. We must use more advanced strategies such as volume by rotation for finding, say, the volume of a figure rotated about the x-axis with the bound $f(x) = 3x^5 + 5x^3$ from 0 to 2.

Let's do something like a Riemann sum. We split the curve into infinitely skinny rectangles. Now, imagine these rectangles are the radius of a circle. Spin this rectangle around the x-axis and you get a circle. Repeat for each rectangle by integrating and using $\mathrm{d}x$.

Let $r = f(x)$. Then, because $A_{circle} = \pi r^2$:

$$\int_0^2 \pi(3x^5 + 5x^3)^2 \, \mathrm{d}x$$

Volume by rotation, washer method

Remember when you took the area between two curves? Now rotate it to form a solid of revolution, but with a hole in the middle. Ta-da.

Volume by rotation about y-axis

To find the volume of a solid by rotating it about the y-axis, you will need to integrate with respect to y or $f(x)$. It is helpful to make x a function of y (i.e. $x = f(y)$). This means if you had the relation $y = \sqrt[3]{x}$ and you wanted to find the volume of the solid of revolution that was rotated about the y-axis from $y = 0$ to $y = 3$, you will need to flip it to solve in terms of x (i.e. $x = f(y) = y^3$), then use this integral:

$$V = \int_0^3 \pi(y^3)^2 \,\mathrm{d}y = \pi \int_0^3 y^6 \,\mathrm{d}y$$

The volume will be ~~YUUUUGE!!!!~~ quite large.

$$V = \pi[\frac{y^7}{7}]\Big|_0^3 = \pi[\frac{3^7}{7}] = \frac{2187}{7}$$

6.4 Volume by shells

Imagine a roll of toilet paper. Volume of shells is asking you to make the strips of the rolls infinitely skinny and count the area like that.

You can also imagine if you rolled out the entire roll of toilet paper. It'll be a very long but flat solid. The important parts here is that:

- The (infinite) skinniness of the toilet paper represents $\mathrm{d}x$.
- $2\pi x$ represents the very long length of the toilet paper (which, if you think about it, when rolled up is the circumference of one of the many circles in the roll).
- Then the width of the toilet paper would be $f(x)$.

The formula for calculating volume by shells is:

$$V = \int_a^b 2\pi x f(x) \,\mathrm{d}x$$

6.5 Length of curves (Arc length)

Have you ever wondered how to calculate the length of a certain curve? Like, actually go along it and calculate the distance? Not making linear approximations?

Let's try $y = x$. From $x = 0$ to $x = 1$, the length will be equivalent to the hypotenuse of the "triangle" formed (where the base is the x-axis from $x = 0$ to $x = 1$ and the height is from $y = 0$ to $y = 1$ at $x = 1$). The length of this, per the 45-45-90 special right triangle rule, is $\sqrt{2}$.

For more complicated cases, we need a formula to determine the length of a curve.

The length of the curve f from a to b will be:

$$L = \int_a^b \sqrt{1 + (f'(x))^2} \,\mathrm{d}x$$

This is just a derivation of the distance formula. We approximate the *linear* distance between infinitely close points x and $x + dx$ on the curve. Then we integrate across the curve from a to b and this is what we get!

A more rigorous proof is indeed available and would not involve approximations.

6.6 Surface area

Knowing how to calculate the area of a region brings us to calculating the volume of a solid somehow defined by said area. Similarly, we will use our newly-gained method of calculating the length of a curve to give us surface areas of solids.

When we rotate a region around a certain axis, this is called a solid of revolution. Its volume is the volume of revolution. In this case, we will be finding the area of a surface of revolution. The formula for finding surface areas of solids of revolution is:

$$S = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\,\mathrm{d}x$$

6.7 Applications in physics

Mass of objects with nonuniform density

Sometimes, the density of objects is uniform. Other times, it just isn't the same from place to place. While the volume is the same (because the space occupied usually doesn't change), the mass will be affected if there is variable or nonuniform density throughout an object.

The classical density equation is:

$$D = \frac{m}{V}$$

Rearranging this to solve for mass, we get

$$m = D \cdot V$$

If mass were homogeneously distributed among an object, we could use the classical equation.

For objects with variable density and therefore heterogeneously distributed mass, we must use calculus to account for every variation.

Suppose a thin bar or wire is represented by $x \in [a, b]$ and its density function is ρ. Then the mass of the object is:

$$m = \int_a^b \rho(x)\,\mathrm{d}x$$

Work

Work is defined as force multiplied by distance. The amount of work you do depends on how much force you exert over a distance.

$$W = Fd$$

$$\text{work} = \text{force} \cdot \text{distance}$$

This base equation assumes that throughout the entire distance work is being applied, the force is constant. However, if force varied throughout this distance, we would need to individually calculate work being done at every variation of force throughout a certain distance.

Let the distance be defined as $b - a$, where a is the starting point and b is the ending point. (For the purpose of this chapter, we will deal with one-dimensional, linear movement, or simple introductory kinematics.) Then, if there is a function that defines the force exerted at a certain distance $x \in [a, b]$, the equation to determine work is:

$$W = \int_a^b F(x)\,\mathrm{d}x$$

Springs (the "bouncy" ones) can move. They start at a regular position (equilibrium). They can stretch and compress, changing its length and therefore its distance from its "equilibrium" position changes. "**Hooke's law** says the force requires to keep the spring in a compressed or stretched position is $F(x) = kx$, where the positive spring constant k measures the stiffness of the spring" (Briggs Cochran). $k > 0$. If $F > 0$, then the spring is stretched, but if $F < 0$, then the spring is compressed. Why? Well:

- If the spring is exerting force, it would exert outwards towards its surroundings and stretch itself. (Think of the spring "pushing outwards." Or, imagine if you were trapped in a box. Your natural instinct would be to push out and to end your claustrophobia, so you can earn freedom.) If the spring itself is exerting force, force is positive.

- However, if the spring has force exerted upon it, then it is its surroundings pushing the spring and therefore compressing it. Forces exerted upon the spring mean the spring is being compressed, and therefore force is negative.

In other terms, Hooke's law says that the *rate* of change of force is linear. Integrating the rate function gives you the work being exerted.

Lifting problems

Work required to lift water:

$$W = \int_a^b \rho g A(y) D(y)\,\mathrm{d}x$$

Hydrostatic forces

Hydrostatic pressure is the pressure given by water when it is at rest, like in a lake. It has the same magnitude in all directions, and is therefore a special type of lifting problem. So for any depth h of water, ρ and g would be the same.

Part II

Calculus 2

Chapter 7

Late transcendentals

Welcome to Calculus 2. Before we get started on the rest of this guide, note that most of Calculus 2 emphasizes memorization, not concepts (which was Calculus 1). Part VII: Late transcendentals is still mainly conceptual, but you will find that from Part VIII: Integration techniques and onwards that memorization is key.

This guide was developed following the progression of Briggs Cochran *Calculus*, 2nd ed. closely, and therefore reflects its late transcendentals pedagogical method.

This guide first introduces inverse functions and then the logarithmic and exponential functions. Then, inverse trigonometric functions will be covered.

7.1 Inverse functions

A function takes in input, does something with it, and outputs the result. Inverse functions does the reverse of the original function.

Finding inverse functions

Let there be a function

$$f(x) = 3x + 1.$$

When you plug in values, it's first multiplied by 3, then 1 is added, and that's the result.

Inverse functions reverse the operations and their order. Currently, $f(x)$ is being represented by manipulating x. Using algebraic rearrangement, we let x be represented by manipulating $f(x)$.

$$f(x) - 1 = 3x$$

$$\frac{f(x) - 1}{3} = x$$

In the original function, we have x and $f(x)$, but we can replace x with $f^{-1}(x)$ and replace $f(x)$ with x. This is also the same thing as

$$\frac{x - 1}{3} = f^{-1}(x).$$

One-to-one

But not just any function has an inverse function. Remember that all functions must have only one x-coordinate corresponding to a y-coordinate, and consequently must pass the vertical line test. Therefore, all inverse functions must also have only one x-coordinate corresponding to a y-coordinate, which means the original function must also have only one y-coordinate corresponding to an x-coordinate.

To clarify the previous paragraph: given the ordered pairs $(1, 7)$, $(3, 4)$, $(5, 7)$, we can say all x-coordinates are unique and therefore it's a function. But if we take the inverse of this function, we get the ordered pairs $(7, 1)$, $(4, 3)$, $(7, 5)$. This won't be a function because 7 corresponds to two y-coordinates: 1 and 5.

Only functions that have unique x and y-coordinates can have inverse functions. In other words, each value in the domain must correspond to exactly one value in the range (Briggs Cochran). One-to-one functions pass both the vertical and horizontal line tests. Only one-to-one functions can have inverse functions.

Domain and ranges of inverse functions

The domain of function f is the range of function f^{-1}. Conversely, the range of function f is also the domain of function f^{-1}.

Testing whether two functions are inverses of each other

Going back to our example functions:

$$f(x) = 3x + 1$$
$$f^{-1}(x) = \frac{x-1}{3}$$

Given two functions $f(x)$ and $g(x)$, if you compose them with each other, they should return the parent linear function ($y = x$).

Let's try it out on our example functions. In this case, we will let $g(x) = f^{-1}(x)$.

$$f(g(x)) = (f \circ g)(x) = 3(\frac{x-1}{3}) + 1 = x - 1 + 1 = x.$$

It did return the parent linear function $y = x$. Those cancellations work beautifully!

Graphing inverse functions

It turns out that when you graph inverse functions, they will be reflections of each other when reflected over the line $y = x$. Go try it on Desmos with our example functions if you want.

Differentiating inverse functions

Since functions inverse to each other reflect over the line $y = x$, differentiating inverse functions is very easy. When you reflect a function over $y = x$, the slope of the reflection is reciprocal to that of the original function. (Not opposite reciprocal, just reciprocal.) This means we can say:

$$(f^{-1})'(y_c) = \frac{1}{f'(x_c)}, \text{ where } y_c = f(x_c).$$

Here's an example (from Briggs Cochran *Calculus*). Say we start out with $y = mx + b$, where $m \neq 0$ and b is the y-intercept. If this were a function, we'd write $f(x) = mx + b$. We know that $f'(x) = m$. To get the inverse, let's solve for x:

$$y - b = mx$$

$$\frac{y}{m} - \frac{b}{m} = x$$

So the inverse function is

$$f^{-1}(x) = \frac{x}{m} - \frac{b}{m}$$

Let's take the derivative. We get

$$(f^{-1})'(x) = \frac{1}{m}$$

(How did we get there? $\frac{b}{m}$ is constant, so that goes away. Furthermore, $\frac{x}{m}$ is equivalent to $\frac{1}{m} \cdot x$, so $\frac{d}{dx}(x \cdot \frac{1}{m}) = \frac{1}{m}$.)

7.2 Natural logarithmic and exponential functions

Remember natural log and e? If not, go review them.

Okay, so if you know what they are, let's go into depth on their definitions and relation to calculus. Turns out that it goes deep.

First off, both the logarithm function and the exponential function are one-to-one. Additionally, given a base b, the logarithm function, when reflected over $y = x$, produces the exponential function of the same base. Therefore, the logarithmic function of a base b is inverse to the exponential function also of base b.

We denote this with:

$$y = b^x \iff x = \log_b y$$

(Note that $\iff$ is a symbol meaning "if and only if.")

Definition of the natural logarithm

The natural logarithm, $\ln$, is defined to be:

$$\ln x = \int_1^x \frac{1}{t}\, \mathrm{d}t$$

We can put the $\mathrm{d}t$ part on the top of the fraction.

$$\ln x = \int_1^x \frac{\mathrm{d}t}{t}$$

Basic properties of the natural logarithm

The natural logarithm is defined by the above integral. That integrand is undefined at $x = 0$.

Furthermore, notice that the lower limit of integration is 1. When $0 < x < 1$, $\ln x < 0$ (is negative). This is because of this integral identity: $\int_a^b f(t)\,\mathrm{d}t = -\int_b^a f(t)\,\mathrm{d}t$, and by convention, we usually like to have $a < b$. If not, then we use that identity to determine that the integral is in fact negative when $x < 1$.

Differentiation and integration of the natural logarithm

It turns out that the natural logarithm's derivative can be found through the definition we were just given. We'll just use the First Fundamental Theorem of Calculus to define the derivative of the natural logarithm.

$$\frac{d}{dx}(\ln x) = \frac{d}{dx}\int_1^x \frac{1}{t}\,\mathrm{d}t = \frac{1}{x} \text{ for } x > 0.$$

Using the chain rule (Briggs Cochran), we find that for all nonzero numbers (now we're covering the negatives), that

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}, x \in \mathbb{R}.$$

also by the chain rule (Briggs Cochran), we find the derivative of the general form of $\ln$.

$$\frac{d}{dx}(\ln|u(x)|) = \frac{d}{du}(\ln|u(x)|)u'(x)$$

$$\frac{d}{dx}(\ln|u(x)|) = \frac{u'(x)}{u(x)}$$

Similarly, we find that the general integral of the inverse function is the natural logarithm of the absolute value:

$$\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + C.$$

Proving logarithm properties with $\ln$'s integral definition

The interesting log properties that you learned in Algebra 2 and Precalculus can be proven using the integral definition:

Product property:

$$\ln xy = \ln x + \ln y \text{ for } x > 0, y > 0$$

Proof:

$$\ln xy = \int_1^{xy} \frac{\mathrm{d}t}{t}$$

$$= \int_1^x \frac{\mathrm{d}t}{t} + \int_x^{xy} \frac{\mathrm{d}t}{t} \text{ (additive property of integrals)}$$

$$= \int_1^x \frac{\mathrm{d}t}{t} + \int_1^y \frac{\mathrm{d}u}{u} \text{ (substitution of } u = \frac{t}{x}\text{)}$$

$$= \ln x + \ln y$$

Quotient property:

$$\ln \frac{x}{y} = \ln x - \ln y \text{ for } x > 0, y > 0$$

Proof:

$$\ln x = \ln\left(y \cdot \frac{x}{y}\right) = \ln y + \ln \frac{x}{y}$$

$$\ln x = \ln y + \ln \frac{x}{y}$$

$$\ln \frac{x}{y} = \ln x - \ln y$$

Power property:

$$\ln x^p = p \ln x \text{ for } x > 0, y > 0$$

Proof:

$$\ln x^p = \int_1^{x^p} \frac{\mathrm{d}t}{t}$$

$$= p \int_1^x \frac{\mathrm{d}u}{u} \text{ (Let } t = u^p;\ \mathrm{d}t = pu^{p-1}\,\mathrm{d}u.\text{)}$$

$$\ln x^p = p \ln x$$

Definition of e

What is the value of x in $\ln x = 1$?

It is a unique value that can be defined through the natural logarithm's integral definition.

$$\ln e = \int_1^e \frac{\mathrm{d}t}{t} = 1.$$

Properties of e^x

These are derived from the proofs of the natural logarithm's properties. (They are also supposed to be trivial review from Algebra II.)

$$e^{x+y} = e^x e^y$$
$$e^{x-y} = \frac{e^x}{e^y}$$
$$(e^x)^p = e^{xp}, x \in \mathbb{Q}$$
$$\ln(e^x) = x, \text{ for all } x$$
$$e^{\ln x} = x, \text{ for } x > 0$$

General exponential functions

We can define general exponential functions in terms of the natural logarithm and e to the x.

$$b^x = (e^{\ln b})^x = e^{x \ln b}$$

So as long as $b > 0, b \in \mathbb{R}, b \neq 1$, then for all real x:

$$b^x = e^{x \ln b}$$

Derivatives and integrals of e^x

Well, this one's kind of strange. Or very interesting.

$$\frac{d}{dx} e^x = e^x.$$

Why is that? Remember e is special. We'll prove this by using the natural logarithm's differentiation rule.

$$\text{Let } u(x) = e^x$$

$$\text{(Remember that } \frac{d}{dx}(\ln u(x)) = \frac{u'(x)}{u(x)}\text{)}$$

So this means that:

$$\frac{d}{dx}(\ln e^x) = \frac{d}{dx}(\ln u(x)) = \frac{u'(x)}{u(x)} = \frac{(e^x)'}{e^x}$$

We also know that $\frac{d}{dx}(\ln e^x) = \frac{d}{dx} x = 1$. So:

$$1 = \frac{d}{dx} x = \frac{d}{dx}(\ln e^x) = \frac{d}{dx}(\ln u(x)) = \frac{u'(x)}{u(x)} = \frac{(e^x)'}{e^x} = 1.$$

$$\text{Therefore, } \frac{(e^x)'}{e^x} = 1.$$

This says that the derivative of e^x divided by e^x is one. Only a value divided by itself is equal to one. That means $(e^x)' = e^x$.

If we want to algebraically represent this, multiply both sides by the denominator, e^x, and you get the **derivative of** e^x:

$$(e^x)' = e^x \cdot 1 \implies \frac{d}{dx}e^x = e^x$$

So the **integral of** e^x follows quite trivially:

$$\int e^x \, \mathrm{d}x = e^x + C$$

If you use the chain rule, we get the following **derivative and integral rules for** e^x:

$$\frac{d}{dx}e^{u(x)} = e^{u(x)}u'(x); \int e^{u(x)}u'(x) = e^{u(x)} + C$$

7.3 Exponential models

There are two kinds of exponential models: growth and decay. They both have the general form:

$$y(t) = Ce^{kt}$$

Relative constant growth

The rate of change of function y is found by taking its derivative:

$$y'(t) = \frac{d}{dt}(Ce^{kt}) = Cke^{kt} = k(Ce^{kt})$$

Notice that if the derivative of function y is itself times a constant k, then the rate of change is proportional to its value.

$$y'(t) = k \cdot y(t)$$

That means we will call $y'(t)$ the **growth rate**. Since it varies by k, it is called *relatively* constant.

Relation to b^x

Since $a^b = e^{b \ln a}$, admittedly $e^{t \ln(a)} = a^t$. Sometimes, it may be more appropriate to represent these functions as $C(r)^t$ instead of $Ce^{t \ln(r)}$. ($r = e^k; k = \ln r$)

7.4 Inverse trigonometric functions

It turns out that the derivatives of inverse trigonometric functions aren't trigonometric.

Derivatives of trigonometric functions:

arcsin:

$$\frac{d}{dx}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}}, \text{ for } |x| < 1 (-1 < x < 1)$$

arccos:

$$\frac{d}{dx}(\arccos(x)) = \frac{-1}{\sqrt{1-x^2}}, \text{ for } |x| < 1 (-1 < x < 1)$$

arctan:

$$\frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2}, \text{ for } -\infty < x < \infty$$

arccot:

$$\frac{d}{dx}(\text{arccot}(x)) = \frac{-1}{1+x^2}, \text{ for } -\infty < x < \infty$$

arcsec:

$$\frac{d}{dx}(\text{arcsec}(x)) = \frac{1}{|x|\sqrt{x^2-1}}, \text{ for } |x| > 1(-\infty < x < 1 \cup 1 < x < \infty)$$

arccsc:

$$\frac{d}{dx}(\text{arccsc}(x)) = \frac{-1}{|x|\sqrt{x^2-1}}, \text{ for } |x| > 1(-\infty < x < 1 \cup 1 < x < \infty)$$

7.5 L'Hôpital's rule for exponentials and comparing function growth rates

L'Hôpital's rule is extended for exponentials and we now know how to use it to compare growth rates.

L'Hôpital's rule for exponentials

When evaluating limits, you are able to simply evaluate the limit of the exponent when the function is of the form e^x.

For example, $\lim_{x\to\infty} e^{\frac{x^3}{2x^4}} = e^{\lim_{x\to\infty} \frac{x^3}{2x^4}}$. You can also let the exponent be known as L, evaluate the limit, assign that value to L, and evaluate e^L for your final answer.

Comparing growth rates

You can measure the growth rate by taking the limit to infinity of dividing one function by another. If the limit evaluates to 0, the bottom one is bigger. If the limit evaluates to infinity (∞), the top one is bigger. If it's between 0 and ∞, then the growth rates are considered comparable.

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \infty; e^x \text{ grows faster than } x^2.$$

$$\lim_{x\to\infty} \frac{\ln(1000x)}{\ln(x)} = 1; e^x \text{ grows at a comparable rate as } x^2.$$

$$\lim_{x\to\infty} \frac{x^2}{\ln(x)} = 0; e^x \text{ grows slower than } x^2.$$

7.6 Hyperbolic functions

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

Chapter 8

Integration techniques

Now we're going to memorize some ways to integrate things in a new way or in an easier way.

8.1 Basic techniques

You can use these basic algebraic massage techniques to integrate:

1. Substitution/u-sub/change of variables
2. Multiplying by 1
3. Splitting or combining fractions
4. Division by rational functions
5. Completing the square

8.2 Integration by parts

Integration by parts is a handy way to evaluate two functions that multiply by each other.

Given that f and g are two differentiable functions, the product rule states that:

$$(fg)' = (f)'g + f(g)'$$

Integrate both sides.

$$fg = \int f'g + \int fg'$$

Rearrange.

$$\int fg' = fg - \int f'g$$

Most calculus literature will represent the equation for integration by parts as:

$$\int u \, \mathrm{d}v = uv - \int v \, \mathrm{d}u$$

While this is somewhat more accurate, this notation is extremely confusing. u and v look very similar, so you can't remember what goes where. Besides, f and g are more familiar with us and it is more helpful to think of the "parts" as functions.

To determine which function is f and which is g' in the original integral, use the LIPET acronym (also known as LIATE). Between your two functions, whichever one meets the criteria first should be f.

1. **L**ogarithms
2. **I**nverse trig functions
3. **P**olynomials
4. **E**xponential functions
5. **T**rigonometric functions

Simple example of integration by parts

$$\int xe^x \,\mathrm{d}x$$

Using LIPET, we hit polynomials first, so x will be f while e^x will be g'.

$$= xe^x - \int e^x \,\mathrm{d}x = xe^x - e^x + C$$

Chained example

$$\int x^2 e^x \,\mathrm{d}x$$

Using LIPET, x^2 will be f and e^x will be g'.

$$= x^2e^x - \int 2xe^x \,\mathrm{d}x = x^2e^x - 2[\int xe^x \,\mathrm{d}x]$$

Now, replace $\int xe^x \,\mathrm{d}x$ with its result from the previous example.

$$= x^2e^x - 2[xe^x - e^x] = e^x(x^2 - 2x + 2) + C$$

Getting clever

$$\int \arctan x \,\mathrm{d}x$$

On first glance, this doesn't look like an integration by parts problem because it only has one function. However, we don't know how to integrate $\arctan$ any other way. If you make $g' = 1$, then you'll see how you can use integration by parts to solve this problem.

$$\int 1 \cdot \arctan x \,\mathrm{d}x = x \arctan x - \int x \cdot \frac{1}{1+x^2} \,\mathrm{d}x$$

The latter integral is actually an $\frac{f'}{f}$ problem.

$$= x\arctan x - \frac{1}{2}\int \frac{2x}{1+x^2}\,\mathrm{d}x = x\arctan x - \frac{1}{2}\ln\left(1+x^2\right) + C$$

Integration by parts for definite integrals

$$\int_a^b fg' = fg\Big|_a^b - \int_a^b f'g$$

8.3 Trigonometric integrals

See another source for integrating trigonometric functions to a certain power.

If you are interested in learning reduction formulas, these will usually either be given to you or they are easily derivable using integration by parts.

$$\int \tan x\,\mathrm{d}x = -\ln|\cos x| + C = \ln|\sec x| + C$$

$$\int \sec x\,\mathrm{d}x = \ln|\sec x + \tan x| + C$$

$$\int \cot x\,\mathrm{d}x = \ln|\sin x| + C$$

$$\int \csc x\,\mathrm{d}x = -\ln|\csc x + \cot x| + C$$

8.4 Trigonometric substitution

Trigonometric sub (trig sub) is useful when you see a square root and a fraction. Use trig identities to make decisions on what trig function you should substitute in.

- $\sin$: $1 - x^2$ or $a^2 - x^2$ $(1 - \sin^2\theta = \cos^2\theta)$
- $\tan$: $x^2 + 1$ or $x^2 + a^2$ $(\tan^2\theta + 1 = \sec^2\theta)$
- $\sec$: $x^2 - 1$ or $x^2 - a^2$ $(\sec^2\theta - 1 = \tan^2\theta)$

Use a triangle to help with converting back from θ to x once you're done integrating.

Example of trig sub

$$\int \frac{x^2}{\sqrt{9-x^2}}\,\mathrm{d}x$$

Let $x = 3\sin\theta$, $\mathrm{d}x = 3\cos\theta\,\mathrm{d}\theta$

$$\int \frac{9\sin^2\theta}{\sqrt{9-9\sin^2\theta}}\cdot 3\cos\theta\,\mathrm{d}\theta = 9\int \frac{\sin^2\theta}{\cancel{\cos\theta}}\cdot\cancel{\cos\theta}\,\mathrm{d}\theta = 9\int \frac{1-\cos(2\theta)}{2}\,\mathrm{d}\theta = 9[\frac{1}{2}\theta - \frac{2\sin\theta\cos\theta}{4}]$$

$$= \frac{9 \arcsin \frac{x}{3}}{2} - \frac{x\sqrt{9-x^2}}{2}$$

There might be errors in this work process but I wrote this at midnight after the semester ended.

8.5 Partial fractions

We can combine fractions that have different denominators into a single fraction with a single denominator. Partial fraction decomposition is the reverse of the process. That's because it's easier to integrate multiple fractions, each with their own denominator, rather than one fraction with a complex denominator.

Traditional method

The traditional method of partial fraction decomposition involves first "splitting" the denominators, then solving for the numerators of each fraction using a system of equations. There are some rules about solving for the numerator.

- If the degree of the denominator is 1 (aka linear function), the numerator will simply be a variable number (e.g. A)
- If the degree of the denominator is 2 (aka quadratic function), the numerator will be in the form $Ax + B$.
- If the denominator is a repeated linear factor (e.g. $(x-3)^2$), then you should make two fractions for this. The first will be, in the case of $(x-3)^2$, $\frac{A}{x-3}$. The second will be $\frac{Bx+C}{(x-3)^2}$.

Don't worry, it isn't as hard as it sounds. Below is an example.

Simple partial fractions example

$$\frac{3}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

Now, multiply by $x(x+1)$.

$$3 = A(x+1) + Bx$$
$$3 = Ax + A + Bx$$
$$0x + 3 = (A+B)x + A$$
$$A = 3$$
$$0x = (3+B)x$$
$$B = -3$$
$$\frac{3}{x(x+1)} = \frac{3}{x} + \frac{-3}{x+1}$$
$$\int \frac{3}{x(x+1)}\,\mathrm{d}x = \int \frac{3}{x}\,\mathrm{d}x - \int \frac{3}{x+1}\,\mathrm{d}x$$
$$= 3\ln x - 3\ln(x+1)$$

Quick evaluation method

Especially for simple partial fraction decompositions, you don't have to multiply both sides by the common denominator, establish a system of equations, etc.; instead, you can let each side go to a certain number and get the value from that. Using the previous example to demonstrate:

$$\frac{3}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

Let $x \to 0$:

$$\frac{3}{(0+1)} = 3 = A$$

Let $x \to -1$:

$$\frac{3}{-1} = -3 = B$$

Basically, use the left side of the equation only. For each decomposed fraction, take away the term that makes it go to zero. Then, plug in the number that makes that decomposed fraction's denominator go to zero.

8.6 Improper integrals

Whenever either of your limits of integration (a and b in $\int_a^b f(x)\,\mathrm{d}x$) go to infinity or any point between the two limits of integration does not exist, you have an improper integral.

The solution to these problems are actually quite simple. From now on, whenever you apply the Second Fundamental Theorem of Calculus ($\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a)$), you should evaluate the limit as the integrated function approaches the limits of integration. That way, you can actually evaluate integrals with ∞! Also, if there is a point (p) between the limits of integration (a and b) that doesn't exist, split the integrals up into $\int_a^p f(x)\,\mathrm{d}x + \int_p^b f(x)\,\mathrm{d}x$. Don't worry about that point not existing when evaluating its integrated function, because remember, we take limits now.

$$\int\limits_a^b f'(x)\,\mathrm{d}x = \lim_{M\to b} f(M) - \lim_{N\to a} f(N)$$

Simple example of improper integral

$$\int\limits_1^\infty \frac{1}{x^2}\,\mathrm{d}x = -\frac{1}{\infty} - (-\frac{1}{1}) = -0 + 1 = 1.$$

8.7 Introduction to differential equations

We will study two forms of differential equations. This is basically finding the integral of an implicit derivative.

Note that $y' = \frac{\mathrm{d}y}{\mathrm{d}x} = y'(x)$.

Separable differential equations

If you can write an equation in the form $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y)$, this is considered a separable differential equation. To evaluate these, rearrange the equation such that $f(x)$ is on the same side as $\mathrm{d}x$ and $g(y)$ is on the same side as $\mathrm{d}y$. Then, integrate both sides and add a C to the right side.

Simple example of separable differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = e^x y$$

$$\frac{1}{y}\,\mathrm{d}y = e^x\,\mathrm{d}x$$

$$\int \frac{\mathrm{d}y}{y} = \int e^x\,\mathrm{d}x$$

$$\ln y = e^x + C$$

Chapter 9

Sequences and series

In precalculus, you learned some basics how sequences and series worked. Some parts may be review for you, but we will be using limits and integrals to evaluate whether an infinite series converges. Essentially, these convergence tests will be the only new material covered here. Therefore, basic information on sequences and series will simply be a review.

9.1 Sequences

A sequence (usually denoted symbolically as $\{a_n\}$) is an ordered list of numbers. Usually, you'll see it in this form:

$$\{a_1, a_2, a_3, \cdots, a_n, \cdots\}$$

Basic sequences are primarily divided into **arithmetic** (air-ith-meh-tic) and **geometric** sequences. Adjacent terms (e.g. a_3 and a_4) in arithmetic sequences change by a common **difference**, whereas adjacent terms in geometric sequences change by a common **ratio**.

If they're not written out for you, there's two ways sequences can be denoted.

The first is a **recursive formula** (also known as a **recurrence relation**). They will be in the form $a_{n+1} = f(a_n)$. For arithmetic sequences, this looks like $a_{n+1} = a_n + d$, where d is the common difference that is being added to/subtracted from the previous term to make the next term. For geometric sequences, this looks like $a_{n+1} = a_n \cdot r$, where r is the common ratio that is being multiplied/divided by the previous term to make the next term.

The second way is through an **explicit formula**, which is in the form $a_n = f(n)$. For arithmetic sequences, this looks like $a_n = d(n-1) + a_1$, which is similar to $y = mx + b$. For geometric sequences, this looks like $a_n = a_1(r)^{n-1}$, similar to $y = b^x$.

Sequence terminology

For a sequence a_n:

- **increasing** - $a_{n+1} > a_n$ (decreasing is the opposite sign, $<$)
- **nonincreasing** - $a_{n+1} \leq a_n$ (nondecreasing is the opposite sign, $\geq$)
- **monotonic** - either nonincreasing or nondecreasing - can't be both (the opposite would be **oscillating**, and a sequence can be neither monotonic nor oscillating)
- **bounded** - if there's a bound M such that $|a_n| \leq M$.

Limit of a sequence

Using limits, we can determine whether a sequence converges, or gets closer to a point L as the sequence approaches ∞. Basically, we're seeing what $\lim_{n\to\infty} a_n$ is equal to. Use previous limit knowledge to evaluate this. If the sequence does not converge to a specific numeric value, then it diverges.

A formal definition of this is done with epsilon-delta.

Convergence of a geometric sequence

Geometric sequences will actually converge if the common ratio r is greater than -1 and less than 1 (i.e. $|r| < 1$ or $-1 < r < 1$). For converging geometric sequences, $f(x) \to 0$ as $x \to \infty$. Later, we will discuss what value a convergent geometric series converges to.

Some other rules on sequence convergence

- The **squeeze theorem** applies to sequences as well. As long as the upper and lower bounding sequences converge to some limit L as $x \to \infty$, then this squeezed sequence must converge.

- All **bounded monotonic sequences** will converge.

9.2 Series

A series is the sum of all of the numbers in an infinite sequence.

For the sequence $\{a_1, a_2, a_3, \cdots\}$, its series is:

$$a_1 + a_2 + a_3 + \cdots = \sum_{k=1}^{+\infty} a_k$$

Series are essentially infinite partial sums, and in fact, series are also known as infinite series. A partial sum is the sum of numbers from a_1 to a_m, where m is a number less than infinity. The sequence of partial sums $\{S_n\}$ is of particular importance; if these approach a limit L as this sequence goes to infinity, then the infinite series converges to this limit L.

A sequence of partial sums looks like this:

$S_1 = a_1$
$S_2 = a_1 + a_2$
$S_3 = a_1 + a_2 + a_3$
$S_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k$

So if $\{S_n\} \to L$ as $n \to +\infty$,

$$\sum_{k=1}^{+\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} S_n = L.$$

Geometric series

The nth partial sum of a geometric sequence can be calculated with this familiar formula:

$$\sum_{k=1}^{n} a_1 r^k = a_1 \frac{1-r^n}{1-r}.$$

But to calculate the value of an infinite geometric series, we must first see if the common ratio r is between -1 and 1 (i.e. $|r| < 1$). If so, then:

$$\sum_{k=1}^{\infty} a_1 r^k = \frac{a_1}{1-r}.$$

Telescoping series

Some special series will feature middle terms that cancel each other out (or as one professor puts it, "mutually annihilates" each other), then the sum of the first and last terms will provide the answer.

$$\sum_{k=1}^{\infty}(\frac{1}{5^k} - \frac{1}{5^{k+1}}) = (\frac{1}{5} - \frac{1}{5^2}) + (\frac{1}{5^2} - \frac{1}{5^3}) + (\frac{1}{5^3} - \frac{1}{5^4}) + \cdots + (\frac{1}{5^n} - \frac{1}{5^{n+1}})$$

$$= (\frac{1}{5} - \cancel{\frac{1}{5^2}}) + (\cancel{\frac{1}{5^2}} - \cancel{\frac{1}{5^3}}) + (\cancel{\frac{1}{5^3}} - \cancel{\frac{1}{5^4}}) + \cdots + (\cancel{\frac{1}{5^n}} - \frac{1}{5^{n+1}}) = \lim_{n\to\infty}(\frac{1}{5} - \frac{1}{5^{\infty+1}}) = \frac{1}{5}.$$

9.3 Properties of convergent series

Suppose $\sum a_k$ converges to A, $\sum b_k$ converges to B, and λ is a real number.

- If $\sum \lambda a_k$ converges to A, $\sum \lambda a_k = \lambda \sum a_k = A$.
- If $\sum a_k$ converges to A and $\sum b_k$ converges to B, $\sum a_k + \sum b_k = \sum(a_k + b_k)$.

9.4 Tests

These tests will help you determine convergence or divergence of series.

Divergence test

This test proves whether a series diverges or not. Note that if a series does not diverge, that does not necessarily mean it will converge. We can only say that it does not diverge.

Divergence test: If $\sum a_k$ converges, then $\lim_{k\to\infty} a_k = 0$. Therefore, if $\lim_{k\to\infty} a_k \neq 0$, then the series diverges.

Integral test

Integral Test (Kallman)
Let $a \in \mathbb{Z}$ and let $f : [a, +\infty) \mapsto [0, +\infty)$ be continuous and nonincreasing. Then the series $\sum_{\ell \geq a} f(\ell)$ converges if and only if

$$\int_a^{+\infty} f(x)\,\mathrm{d}x < +\infty.$$

In fact,

$$\int_a^{+\infty} f(x)\,\mathrm{d}x \leq \sum_{\ell \geq a} f(\ell) \leq \int_a^{\infty} f(x)\,\mathrm{d}x + f(a).$$

(Side note: if and only if can also be written as "iff" (without quotes) or the symbol $\iff$.)

That being said, the integral of $f(x)$ does not equal the series' value except through pure coincidence!

p-Series

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

This series converges if $p > 1$ and diverges if $p \leq 1$. That's it.

Note that you must determine whether these series converge or not *before* you try to apply these properties!

Ratio test

The ratio test is great for series where you can quantify a_{n+1}.

Let $\sum a_k$ be a positive-termed series and

$$r = \lim_{k \to +\infty} \frac{a_{k+1}}{a_k}$$

- If $0 \leq r < 1$, the series converges.
- If $r > 1$, including when $r = +\infty$, then the series diverges.
- If $r = 1$, the test is inconclusive.

(Briggs Cochran)

Root test

The root test is great for series where you may need roots to be ”cancelled out.”

Let $\sum a_k$ be a nonnegative-termed series and

$$\rho = \lim_{k\to+\infty} \sqrt[k]{a_k}$$

- If $0 \le \rho < 1$, the series converges.
- If $\rho > 1$, including when $\rho = +\infty$, the series diverges.
- If $\rho = 1$, the test is inconclusive.

(Briggs Cochran)

Limit comparison test

For very similar series, we can use the limit comparison test to determine the convergence or divergence of two series. Usually, this is best used when you know whether series $\sum b_k$ converges or diverges, so you can determine whether the similar series $\sum a_k$ converges or diverges.

Let $\sum a_k$ and $\sum b_k$ be series with positive terms and let

$$\lim_{k\to\infty} \frac{a_k}{b_k} = L.$$

- If $0 < L < +\infty$, then $\sum a_k$ and $\sum b_k$ either both converge or both diverge.
- If $L = 0$ and $\sum b_k$ converges, then $\sum a_k$ also converges.
- If $L = +\infty$ and $\sum b_k$ diverges, then $\sum a_k$ also diverges.

(Briggs Cochran)

Alternating series test

For series whose succeeding terms switch signs (i.e. $+$ to $-$ or $-$ to $+$), these series are considered **alternating series**. An example would be the *alternating harmonic series*. The regular harmonic series:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n}$$

diverges. However, the alternating harmonic series:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots + \frac{1}{n}$$

converges.

In sigma notation, we can represent this flipping of signs by $(-1)^k$ or $(-1)^{k+1}$.

To determine whether an alternating series in fact converges, it must meet two requirements.

1. All terms must be decreasing in magnitude. Graphically, this means the terms will go up and down but less so as it approaches $+\infty$.
2. $$\lim_{k\to+\infty} a_k = 0.$$

While evaluating this limit, you can remove the $(-1)^k$ term from the overall series expression, as it will have no effect on the limit.

9.5 Absolute and conditional convergence

It turns out that if for a series $\sum a_k$, if $\sum |a_k|$ converges, then $\sum a_k$ must also converge. This is called **absolute convergence**. However, if $\sum |a_k|$ diverges, we cannot say that $\sum a_k$ will necessarily diverge. In fact, when $\sum a_k$ converges, this is called **conditional convergence**.

Chapter 10

Power series

A power series is an infinite series whose terms include powers of a variable.

These power series, when expressed to a certain degree, can provide approximations. There are also ways to write entire functions in terms of power series.

Here is a general power series:

$$\sum_{k=0}^{+\infty} c_k(x-a)^k = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n + c_{n+1}(x-a)^{n+1} + \cdots$$

10.1 Taylor series and Maclaurin series

Taylor series are a handy way of approximating functions in terms of polynomials. We use these since it's easier to deal with polynomials when using calculus.

Taylor series will center at a specific point on the function's x-axis (let's call it a). Taylor series are a valid approximation of a certain function at that point a and points close to it. As you get further away from a, the less valid a Taylor series will get to approximating its original function.

A Maclaurin series is simply a Taylor series with $a = 0$.

Taylor's Formula with Remainder

Let $n \in \mathbb{Z}$; $x, a \in \mathbb{R}$; and f be $(n+1)$-times continuously differentiable on the closed interval between a and x. Then,

$$f(x) = \sum_{\ell=0}^{n} \frac{(x-a)^\ell}{\ell!}(D^\ell f)(a) + \frac{1}{n!}\int_a^x (x-t)^n (D^{n+1} f)(t)\,\mathrm{d}t$$

(Kallman)

Taylor series are approximations without its remainder. The integral part is called the remainder or the error term. With the remainder, the sum and integral added together completely equal the original function. Notice that you will iteratively generate derivatives from 0 to n, add them up, and then inside the integral you will evaluate the $n+1$th derivative.

Taylor polynomials

Since polynomials will eventually have a derivative of zero, this will affect the remainder term: it becomes zero. For example, we have a fourth-degree polynomial. If $n = 4$, then the fifth derivative of this polynomial would be zero and cancels out the remainder/error term. If the error term is exactly zero, this means the Taylor series without the integral is equal to the original function.

Notable Taylor expansions

You will likely need to know these expansions and for what range they are valid.

$$e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots, x \in \mathbb{R}$$

$$\cos x = \sum_{n=0}^{+\infty} \left((-1)^n \frac{x^{2n}}{(2n)!}\right) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots, x \in \mathbb{R}$$

$$\sin x = \sum_{n=0}^{+\infty} \left((-1)^n \frac{x^{2n+1}}{(2n+1)!}\right) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots, x \in \mathbb{R}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \cdots, x \in (-1, 1]$$

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} + \cdots, x \in [-1, 1]$$

Part III

Calculus 3: Multivariable Calculus

Chapter 11

Vectors, vector functions, and vector function calculus

We begin Multivariable Calculus (Calculus 3) with vectors. Be sure you remember polar and parametric functions and conic sections from Precalculus.

11.1 Introduction to vectors

Vectors are quantities that have both a direction and a magnitude. In contrast, scalars only have a magnitude. We depict vectors as line segments with an arrow on the vector's head and a dot on the vector's tail.

Magnitude is always positive because it is length, and length cannot be negative. In fact, that's why we use directions: to denote the direction. So instead of using + or - for $0°$ and $180°$, we can simply use the direction to represent any angle we want! Also, since magnitude is always positive, we use the absolute value notation to denote the magnitude of a vector. Say our vector is $\vec{v}$. Then, our magnitude is denoted by $|\vec{v}|$.

There are several kinds of special vector forms that we use as building blocks to make more complex vectors:

- We call a vector positioned with its tail at the origin a **position vector**. It's equal to the other vectors with the same magnitude and direction, just put at the origin to make life easier.

- A **unit vector** is a vector that has a magnitude of 1 but can be in any direction.

- A **zero vector** has no direction and a magnitude of 0. It can be resolved to the components $\langle 0, 0 \rangle$. Note that there is a difference between the scalar number 0 and a zero vector!!

Vectors are denoted either by its magnitude and direction (denoted by $|\vec{TH}|$ for magnitude, and if you really want, $\theta_{\vec{TH}}$ for direction, but there isn't an established notation for the direction of a vector) or it can be resolved to its orthogonal components. If you think about it, magnitude and direction is kind of like (r, θ) of polar coordinates. Let's convert this to rectangular Cartesian coordinates. This means we can denote the vector by x and y components. The true meaning of these components will be explained later, but for now, we can say a vector $\vec{v}$ with magnitude $|\vec{v}|$ and direction θ is represented in component form as $\langle \vec{v} \cos\theta, \vec{v} \sin\theta \rangle$.

We can obtain vectors from two points on the plane by using the **head minus tail** rule. If the head is denoted by $H(h_x, h_y)$ and the tail is $T(t_x, t_y)$, then the vector is called $\vec{TH}$ and we get the vector in component form $\langle h_x - t_x, h_y - t_y \rangle$.

If any two vectors have the same magnitude and direction, then **they are the same vector**. If two vectors have different magnitudes but the same direction (or 180° opposite) then they are parallel vectors. This also means they are scalar multiples of each other.

Basic vector operations

Multiplying a scalar by a vector is simply multiplying the scalar by the magnitude of the vector to get a new vector. If the scalar is less than 0, then the direction of the vector is flipped, since we can't have negative magnitudes.

To add two vectors, simply place either of the vectors' heads to the other's tail (triangle rule or head-to-tail rule). In this case, we'll call the first vector $\vec{u}$ and the second one $\vec{v}$. Have $\vec{u}$'s head touching $\vec{v}$'s tail, then draw a straight line from the tail of $\vec{u}$ to the head of $\vec{v}$. Then add an arrow on the side of the line segment closest to the head of $\vec{v}$. This straight line is the **resultant vector**.

You can also add vectors using the parallelogram rule. Place both vectors tail-to-tail. From each vector's heads, draw the other vector. This will form a parallelogram. Where they meet is the resultant vector's head, and the tail of the resultant vector is simply where the two vectors' tails met.

To subtract vectors, simply reverse the direction of the second vector (let's say it's $\vec{v}$) and call it $-\vec{v}$. Then subtract like so: $\vec{u} + (-\vec{v})$.

Components

Components in $\langle x, y \rangle$ format can also be written as $x\hat{\imath} + y\hat{\jmath}$. The $\hat{\imath}$ and $\hat{\jmath}$ represent the unit vectors in the positive x and positive y directions of a graph, respectively. So $\hat{\imath} = \langle 1, 0 \rangle$ and $\hat{\jmath} = \langle 0, 1 \rangle$. x and y are scalars, so you just basically multiply the scalars by the unit vectors and add them up. Per the triangle method, you'll get the original vector!

To find the magnitude from the vectors, you can simply do $\sqrt{x^2 + y^2}$, the distance formula. To find the angle in $\mathbb{R}^2$ (meaning 2D space), do $\arctan\left(\frac{y}{x}\right)$.

Application of vectors

You'll see them applied in physics. Relative velocities may be something you will see.

Vectors in 3D

Nothing much changes, but you might want to know that the distance formula in $\mathbb{R}^3$ is now:

$$\sqrt{x^2 + y^2 + z^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Note that we aren't taking the cube root. It's still the square root.

11.2 Introduction to the 3D plane

The 3D plane is very self-explanatory. You add a third z-axis perpendicular to both x and y axes, and consequently, ordered pairs become ordered triples like (x, y, z). Now we have three planes: xy, xz, and yz.

We usually use a right-handed coordinate system, where you start with your fingers (yes, YOUR fingers on your right hand) pointing straight at the x-axis. Then, curl them towards the y-axis. This is how we determine where is x and where is y.

11.3 Vector operations: dot and cross products

There are two main ways to multiply vectors: dot and cross products. The primary difference is that for the dot product, you just multiply each vector within the component, then add the product of the individual components together. Also, the dot product is a scalar. Conversely, the cross product will result in a vector. (Sometimes, dot products will be called scalar products and cross products will be called vector products.)

Dot products

The dot product is one of two ways to multiply two vectors. A dot product is denoted as $\vec{u} \cdot \vec{v}$ and results in a scalar value.

There are two ways to calculate dot products. If you have two vectors $\vec{u}$ and $\vec{v}$ in component form $\langle u_1, u_2, u_3 \rangle$ and $\langle v_1, v_2, v_3 \rangle$, then the dot product is simply $u_1v_1 + u_2v_2 + u_3v_3$.

The other way to find it is if you have the two vectors' magnitudes and the angle between the two vectors, you can calculate the dot product using the following formula:

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos \theta$$

If the dot product of two vectors equals 0, that means $\theta = 90°$ and therefore $\cos(90°) = 0$, so the dot product will be 0. This also works with component form.

Projections

Imagine you have vector $\vec{b}$ flat on the x-axis and $\vec{a}$ pointing up at an angle of $60°$ from $\vec{b}$. Then the "shadow" that $\vec{a}$ casts upon the direction of $\vec{b}$ is called the **projection**.

The projection is found by the following formula:

$$\text{proj}_{\vec{b}}\vec{a} = (\frac{\vec{a} \cdot \vec{b}}{\vec{b} \cdot \vec{b}})\vec{b}$$

$\frac{\vec{a} \cdot \vec{b}}{\vec{b} \cdot \vec{b}}$ determines the magnitude, whereas the exact direction of the projection is kept from $\vec{b}$. You can also find parallel and normal forces by using projections.

Application: work

You can find work by using the definition of dot products.

$$W = |\vec{F}| \cos\theta \, |\vec{d}| = \vec{F} \cdot \vec{d}$$

Note that since the dot product is taken of force and direction, work is a scalar.

Cross products

Cross products are the other way to multiply vectors. They are denoted as $\vec{u} \times \vec{v}$ and results in a vector value.

To find the cross product using components, you take the determinant of a matrix with the top row as $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ in that order. The result is then a vector.

If you have the magnitudes of your two vectors and the angle between them, you can simply use the following formula:

$$\vec{u} \times \vec{v} = |\vec{u}||\vec{v}| \sin\theta$$

Cross products are **anticommunicative**, meaning that $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$. In other words, order matters!!

Note that the cross products results in a vector. This vector's magnitude can be found with the above formula. However, its direction is unclear unless you use the right-hand rule.

Here's how you use the right hand rule for $\vec{u} \times \vec{v}$. Start at the first operand ($\vec{u}$) with your fingers pointed straight. Then curl your fingers towards the second operand ($\vec{v}$). If your thumb points upwards, then the result of the cross product is also upwards. Similarly, if your thumb points downwards, the result is also downwards. This only works for 2D vectors being crossed though. For 3D vectors, you must use the determinant method.

There are different ways to calculate the determinant, so if you do not know how to calculate it, please research it.

Additionally, the magnitude of a cross product is equivalent to the area of the parallelogram made if the two operand vectors were placed tail-to-tail and formed a parallelogram.

11.4 Lines and curves in space

We will begin 3D functions by using vector-valued functions. What are those? You can think of them in two ways, both valid.

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

1. A set of three parametric functions with respect to t that give the position in $\mathbb{R}^3$
2. x, y, and z are with respect to t and are the components of $\vec{r}$

Lines

In 2D, we use the notation $y = mx + b$. Same principle here, except applied parametrically and we use vectors.

We'll have a fixed point that we can use as our "b" and a variable point.

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$$

The above means:

$$\vec{r} = \vec{r_0} + \vec{v}$$

Parametrically, we can split this into three equations:

$$x = x_0 + at$$
$$y = y_0 + bt$$
$$z = z_0 + ct$$

$$\text{for } -\infty < t < \infty$$

Whenever lines need to pass through a certain point, this certain point is the "b", which is now called $\vec{r_0}$.

Curves

What if it isn't straight? Then we'll use three different functions, each representing how much its component changes.

$$\vec{r}(t) = \langle f(t),\, g(t),\, h(t) \rangle = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}$$

Orientation of curves

As these vector-valued functions are parametric, there is a direction, or orientation, in which these functions are graphed. For instance, $\langle \cos t, \sin t \rangle$ will graph the same thing as $\langle \cos t, -\sin t \rangle$ over the range $0 \leq t < 2\pi$, but in a different orientation. Because the second vector's y-component is $-\sin t$, this will affect the orientation. So when $t = \frac{\pi}{2}$, the first vector and the second vector will have gone different ways.

Therefore, the tangent vector, which indicates the direction of the curve at a certain instant of time, will also have the same orientation of the curve at that point.

11.5 Introduction to vector function calculus

Vector function calculus is similar to regular (scalar) calculus. Keep in mind the relationship between vector-valued functions and parametric functions. Be sure to consider each component separately, and then evaluate how the components will act together. This is the guiding principle of vector function calculus.

Limits and continuity

$$\lim_{t \to a} \vec{r}(t) = \vec{L}$$

provided that

$$\lim_{t \to a} |\vec{r}(t) - \vec{L}| = 0.$$

This latter equation is a function of the variable t (i.e. parametric-like). That means we should take the limit of each component of $\vec{r}$ to obtain its limit as it approaches point a.

If

$$\vec{r}(t) = \langle f(t),\, g(t),\, h(t) \rangle$$

This means

$$\lim_{t \to a} \vec{r}(t) = \langle \lim_{t \to a} f(t),\, \lim_{t \to a} g(t),\, \lim_{t \to a} h(t) \rangle$$

Continuity also applies. Remember, if there's a "hole" or a sharp turn (like at $x = 0$ for $y = |x|$), then the function is not continuous at that point. If at t there is a discontinuity in any of the component functions, then the overall function will also be discontinuous at t.

Differentiation of vectors and tangent vector

The formal definition is

$$\vec{r}'(t) = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t}$$

This just translates to the following equation for derivatives:

$$\vec{r}'(t) = f'(t)\hat{\imath} + g'(t)\hat{\jmath} + h'(t)\hat{k}$$

It's very straightforward: you take the derivative of each component's function with respect to t.

The functions $f, g,$ and h need to all be differentiable.

As long as the derivative isn't equal to the zero vector $\vec{0}$, $\vec{r}'(t)$ also serves as the tangent vector at the point corresponding to $\vec{r}(t)$.

The unit tangent vector is $T(t) = \frac{\vec{r}'(t)}{|\vec{(r)}'(t)|}$.

Derivative rules for vectors

These rules are pretty similar to scalar derivative rules.

1. **Constant rule**
$$\frac{d}{dt}(c) = \vec{0}$$

2. **Sum rule**
$$\frac{d}{dt}(\vec{u}(t) + \vec{v}(t)) = \vec{u}(t) + \vec{v}(t)$$

3. **Product rule (one vector and one scalar)**
$$\frac{d}{dt}(f(t)\vec{u}(t)) = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$$

4. **Chain rule**
$$\frac{d}{dt}(\vec{u}(f(t))) = \vec{u}'(f(t))\, f'(t)$$

5. **Dot product rule**
$$\frac{d}{dt}(\vec{u}(t) \cdot \vec{v}(t)) = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$$

6. **Cross product rule (order matters!)**
$$\frac{d}{dt}(\vec{u}(t) \times \vec{v}(t)) = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$$

Indefinite integration of vectors

We'll use the typical definition of integrals as we learned them in Calculus 1.

Let $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ be the derivative of $\vec{R}(t) = \langle F(t), G(t), H(t)\rangle$. Then,

$$\int \vec{r}(t)\,\mathrm{d}t = \vec{R}(t) + \vec{C}$$

$$\int \langle f(t), g(t), h(t)\rangle\,\mathrm{d}t = \langle F(t), G(t), H(t)\rangle + \langle C_1, C_2, C_3\rangle$$

Note that $\vec{C}$ is a vector too, and remains separate from the integrated vector unless we know the values of the components of $\vec{C}$

Definite integration of vectors

When we add the limits of integration a and b, we integrate with respect to t. Again, componentize. The definite integral of $\vec{r}(t)$ on $[a, b]$ is:

$$\int_a^b \vec{r}(t)\,\mathrm{d}t = \Big(\int_a^b f(t)\Big)\hat{\imath} + \Big(\int_a^b g(t)\Big)\hat{\jmath} + \Big(\int_a^b h(t)\Big)\hat{k}$$

11.6 Applications of vector function calculus

3D vector-based kinematics

Kinematics is the study of motion in physics. Given a position function $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$, we can find the velocity function $\vec{v}(t) = \vec{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle$ and the acceleration function $\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t) = \langle v_x'(t), v_y'(t), v_z'(t)\rangle = \langle x''(t), y''(t), z''(t)\rangle$.

Displacement is the distance between the start and end of an object, not the total distance travelled. Therefore, speed is simply the derivative of displacement. Speed is also therefore the magnitude of the velocity vector.

$$|\vec{v}(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$$

Given an acceleration vector and the initial velocity vector, you can find the velocity vector by integrating the acceleration vector and setting the mysterious $\vec{C}$ vector equal to the initial velocity vector. Same principle for finding the position vector given the initial position vector.

Let's say $\vec{a}(t) = \langle 2, 5, 3\rangle$ is the acceleration vector, $\vec{v_0} = \langle 1, 2, 7\rangle$, and $\vec{p_0} = \langle 8, 9, 1\rangle$.

Then the velocity vector can be found like so:

$$\vec{v}(t) = \int \vec{a}(t)\,\mathrm{d}t + \vec{C_v} = \int \vec{a}(t)\,\mathrm{d}t + \vec{v_0}$$

$$\vec{v}(t) = \int \langle 2, 5, 3\rangle\,\mathrm{d}t + \vec{v_0} = \langle 2x + 1, 5y + 2, 3z + 7\rangle$$

And the position vector:

$$\vec{p}(t) = \int \vec{v}(t)\,\mathrm{d}t + \vec{C_p} = \int \vec{v}(t)\,\mathrm{d}t + \vec{p_0}$$

$$\vec{p}(t) = \int \langle 2x+1, 5y+2, 3z+7 \rangle \, \mathrm{d}t + \vec{p_0} = \langle x^2+x+8, \frac{5y^2}{2}+2y+9, \frac{3z^2}{2}+7z+1 \rangle$$

For projectile motion, the acceleration vector $\vec{a}(t) = \langle 0, -9.8, 0 \rangle$, assuming we're using meters.

If there's crosswind and a velocity or acceleration vector is given, simply add the crosswind vector to the existing vectors to correct for crosswind.

Circle position and tangent vector relation

If $\vec{r}(t)$ denotes the path of a curve and $\vec{v}(t)$ is the vector tangent to the line at point t, then if

$$\vec{r}(t) \cdot \vec{v}(t) = 0$$

at all points of t, then $\vec{r}(t)$ denotes a circle. This is because if $|\vec{r}(t)|$, the distance between the curve and a certain point on the line, were always constant, then that would create a circle. From that, if we know that the tangent vector is always orthogonal to the position for all t, then it must be a circle.

Shortest distance from line to point

You can also use projections to find the first leg from the hypotenuse and solve for the shortest distance as the remaining leg of the right triangle using the Pythagorean theorem.

Let S be a point in $\mathbb{R}^3$. Let the line be $\vec{L}$.

Let $\vec{p_0}$ be a vector with components comprised of the respective constants in line $\vec{L}$. Then, let $\vec{v}$ be a vector with components comprised of the respective coefficients of t in line $\vec{L}$.

So, if $\vec{L} = \langle 1+2t, 3+4t, 5+6t \rangle$, then $\vec{p_0} = \langle 1, 3, 5 \rangle$ and $\vec{v} = \langle 2, 4, 6 \rangle$. Also, $\vec{L} = \vec{p_0} + t\vec{v}$.

$\vec{p_0 S}$ is the hypotenuse of the right triangle formed between the shortest distance between point and line and the line from the p_0. The shortest distance can be found by multiplying the magnitude by $\sin\theta$.

$$d = |\vec{p_0 S}| \sin\theta$$

Conveniently, the cross product definition also includes a $\sin\theta$ so we can substitute $\sin\theta$ by using the cross product of $\vec{p_0}$ and $\vec{v}$:

$$|\vec{p_0 S} \times \vec{v}| = |\vec{p_0 S}||\vec{v}| \sin\theta$$

Divide both sides by $|\vec{v}|$.

$$\frac{|\vec{p_0 S} \times \vec{v}|}{|\vec{v}|} = |\vec{p_0 S}| \sin\theta$$

Voilà! You can use this cross product formula to find the shortest distance between point S and line L.

(Side note: order for cross product does not matter since we're only dealing with magnitudes. If we were finding the direction as well, then right hand rule and ordering rules of cross product (i.e. anticommunicative property) applies as usual.)

11.7 Length of vector curves

Vector arc length

In Calculus 1, we used the Mean Value Theorem to derive arc length. Doing something similar but componentized for vectors gets us the vector arc length formula.

Let $\vec{r}(t)$ on $[a, b]$ be the curve segment of which we're trying to find the length.

$$L = \int_a^b |\vec{r}'(t)| = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)}\,\mathrm{d}t$$

Application of vector arc length

If we want to find the distance that an object travelled, then we simply integrate the speed vector function with respect to time.

Let $\vec{r}'(t) = \vec{v}(t)$. Then,

$$L = \int_a^b |\vec{v}(t)|\,\mathrm{d}t$$

Parameterization of arc length

This topic will not be covered.

11.8 Curvature and normal vectors

Curvature

We find the curvature $K(t)$, a scalar, through the following formula, assuming that $\vec{r}(t)$ is a smooth parameterized curve and given that $\vec{T}(t)$ is the unit tangent vector.

$$K(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$$

If we know the velocity and acceleration vectors, then the alternative formula is:

$$K(t) = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}$$

Normal vectors

Normal vectors are orthogonal to tangent vectors. Therefore, use the dot product = 0 rule. A vector $\vec{N}$ is normal to $\vec{T}$ if

$$\vec{T} \cdot \vec{N} = 0.$$

Chapter 12

Functions of several variables

Whereas vector functions had one independent variable and two dependent variables, we'll now learn about functions with multiple independent variables but just one dependent variable.

12.1 Planes and surfaces

In $\mathbb{R}^2$, the simplest curve is a line. So in $\mathbb{R}^3$, the simplest surface is a plane.

Very simple planes include $x = 1$. This means on the x-axis, only at $x = 1$ is there a solution, but the entire yz plane is a solution when at $x = 1$.

Let's say we have a point P_0 on this plane of ours. Well, the set of all points on the plane will be represented by the generic point P. Then, let's say we have a vector normal to this plane called $\vec{n}$. We can find the plane by taking the dot product of $\vec{P_0P}$ and $\vec{n}$, which will equal 0. This gets us our plane's equation.

If $P_0 = (1, 2, -3)$ and $\vec{n} = \langle -4, 2, -7 \rangle$, with $P = (x, y, z)$, then $\vec{P_0P} = \langle x - 1, y - 2, z + 3 \rangle$. Therefore, $\vec{n} \cdot \vec{P_0P} = 1(x + 4) + 2(y - 2) - 3(z + 7) = 0$.

Let $P_0(x_0, y_0, z_0)$ and $\vec{n} = \langle a, b, c \rangle$. Then, the plane is described by the equation:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

or

$$ax + by + cz = d$$

Where $d = ax_0 + by_0 + cz_0$.

Finding the equation of a plane given three points

If you are given any three points that lie on a plane, you can find the plane's equation by finding two tail-to-tail vectors from these three points and taking their cross product to get the normal vector $\vec{n}$. Then, simply use any of the three points as P_0.

Let's say we have points $A(1, 0, 0), B(0, 2, 1), C(0, 0, 3)$. We'll find $\vec{AB} = \langle -1, 2, 1 \rangle$ and $\vec{AC} = \langle -1, 0, 3 \rangle$. Then, find their cross product by taking their determinant:

$$\vec{AB} \times \vec{AC} = \det \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 2 & 1 \\ -1 & 0 & 3 \end{vmatrix} = (6\hat{\imath} - \hat{\jmath}) - (-3\hat{\jmath} - 2\hat{k}) = 6\hat{\imath} + 2\hat{\jmath} + 2\hat{k} = \langle 6, 2, 2 \rangle = \vec{n}$$

Then we simply plug in $\vec{n}$ and one of the points (let's pick A) and get the equation.

$$6x + 2y + 2z = 6 \cdot 1 + 2 \cdot 0 + 2 \cdot 0$$

$$3x + y + z = 3$$

Finding the point at which a line intersects a plane

Lines in $\mathbb{R}^3$ must be written in vector form, which is essentially parametric form. Given the plane's equation, plug in each of the line's components and solve for t. Then, plug that t value back into the line.

For instance, if the plane is represented by $2x + y - 4z = -3$ and the line by $\vec{r}(t) = \langle 3 - 4t, \frac{5}{2} + t, 1 + 2t \rangle$, then we get:

$$x = 3 - 4t$$

$$y = \frac{5}{2} + t$$

$$z = 1 + 2t$$

and we plug in these to the plane equation: $2(3 - 4t) + \frac{5}{2} + t - 4(1 + 2t) = -3$. We get $6 - 8t + \frac{5}{2} + t - 4 - 8t = -3$, which leads us to find that $t = \frac{1}{2}$.

The point of intersection is: $(3 - 4\frac{1}{2}, \frac{5}{2} + \frac{1}{2}, 1 + 2\frac{1}{2}) = (1, 3, 2)$.

Finding the acute angle between two planes

The normal vectors of each planes are orthogonal to their respective planes. Therefore, the angle between two planes can also be found by the angle between their respective normal vectors. In fact, we will use the dot product formula to find the angle between the two normal vectors.

Say we have two equations, A and B.

$$A : x - 2y - 2z = 15$$

$$B : 3x + 4y = 0$$

Their normal vectors are as follows: $\vec{n_A} = \langle 1, -2, -2 \rangle, \vec{n_B} = \langle 3, 4, 0 \rangle$. We can discard the d value because it's the angle we care about, and the constant multiples don't matter. If we set $d = 0$, that would be parallel to $d =$ anything, so long as the left side is the same. Parallel means same angles are kept.

To find the angle, we use the dot product formula $\vec{n_A} \cdot \vec{n_B} = |\vec{n_A}||\vec{n_B}| \cos\theta$ and rearrange it to $\theta = \arccos\left(\frac{\vec{n_A} \cdot \vec{n_B}}{|\vec{n_A}||\vec{n_B}|}\right)$. Now, we plug it in and find the angle.

$$\theta = \arccos\left(\frac{1 \cdot 3 + -2 \cdot 4 + -2 \cdot 0}{\sqrt{1^2 + (-2)^2 + (-2)^2} \cdot \sqrt{3^2 + 4^2 + 0^2}}\right) = \arccos\left(\frac{-1}{3}\right) = -18.435^\circ$$

Since this angle is less than 90°, we take off the negative and we get $\theta = 18.435^\circ$.

Find the line of intersection between two planes

Similar to finding the acute angle between two planes, we use the two planes' normal vectors. We take the cross product between the two normal vectors to get a line parallel to the line of intersection. We then use a system of equations to solve for a point.

Remember that the solution should be in the form $\langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle$.

If our two planes are defined by the equations A: $2x - y + z = 13$ and B: $3x + 2y + z = 7$, then their normal vectors are $\vec{A} = \langle 2, -1, 1\rangle$ and $\vec{B} = \langle 3, 2, 1\rangle$. Now, time for the cross product:

$$\vec{A} \times \vec{B} = \det \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & 1 \\ 3 & 2 & 1 \end{vmatrix} = -\hat{\imath} + 3\hat{\jmath} + 2\hat{k} - (2\hat{\imath} + 2\hat{\jmath} - 3\hat{k}) = -3\hat{\imath} + \hat{\jmath} + 5\hat{k} = \langle -3, 1, 5\rangle$$

It also does not matter which order we take the cross product, because if the cross product went in the opposite direction, the direction doesn't matter and it still is parallel.

Now, we have this cross product. It is equivalent to $\langle a, b, c\rangle$ in our template. Next up is to find the $\vec{p_0}$.

Since this is a line, we can arbitrarily set one of the variables to any value, as long as it is constant and unchanged. For simplicity, we'll set $x = 0$ in the system of equations. You could set $z = 1000$, but that makes life harder.

Let $x = 0$:

$$-(y + z = 13)$$
$$2y + z = 7$$

So $y = -6$, $z = 7 - 2(-6) = 19$, and we already knew $x = 0$. Therefore, $\vec{p_0} = \langle 0, -6, 19\rangle$.

So, the equation of the line is $\langle 0, -6, 19\rangle + t\langle -3, 1, 5\rangle = \langle -3t, -6 + t, 19 + 5t\rangle$.

Find the shortest distance between a point and the plane

The shortest distance between a point (let's call the point S and the shortest distance d) and the plane is perpendicular to the plane. We can use the normal vector, which we already know about, to represent this distance. Then, we'll find some point on the plane and call it P_0; it doesn't matter where it is. $|\vec{P_0S}|$ shall represent the distance between this random point on the plane and the point S. To get the orthogonal shortest distance D, we take the cosine function of θ (i.e. $D = |\vec{P_0S}| \cos\theta$). Except, we don't know what the value of θ is.

Using the definition of dot products, we get:

$$\vec{P_0S} \cdot \vec{n} = |\vec{P_0S}||\vec{n}| \cos\theta$$

Divide both sides by $|\vec{n}|$ and we get

$$D = |\vec{P_0S}| \cos\theta = \frac{\vec{P_0S} \cdot \vec{n}}{|\vec{n}|}$$

A simpler formula can be proven. Given that the equation of a plane is $ax + by + cz = d$ and S to (x_0, y_0, z_0), we get that:

$$D = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

by setting point P to be (x, y, z). Per the definition given above, we get $\vec{P_0S} = \langle x_0 - x, y_0 - y, z_0 - z \rangle$ and therefore

$$D = \frac{\vec{P_0S} \cdot \vec{n}}{|\vec{n}|} = \frac{a(x_0 - x) + b(y_0 - y) + c(z_0 - z)}{\sqrt{a^2 + b^2 + c^2}}$$

After distributing this, we get

$$= \frac{ax_0 + by_0 + cz_0 - (ax + by + cz)}{\sqrt{a^2 + b^2 + c^2}}$$

Replace $ax + by + cz$ with d as specified in our given information of this proof. We therefore get

$$D = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

Cylinders

So far, we've been dealing with flat surfaces that can be described as $z = x + y$, where x and y have their coefficients and stuff. However, this equation isn't a curve (i.e. it's not "wavy"), it's just a plane. On the other hand, we call these wavy surfaces **cylinders**. They're not the same thing as cylinders that you typically think of. They're two totally different things.

The official definition of cylinders is as follows: given a curve $C(x, y)$ in plane P and a line ℓ not in P, a cylinder is the surface consisting of all lines parallel to ℓ that pass through C. (Briggs Cochran)

Cylinders are easy to identify–they will be missing one of the variables: x, y, or z. If we are given the parabola $y = x^2$, in $\mathbb{R}^2$ it's just a regular parabola. However, in $\mathbb{R}^3$, since z is not specified, it is always true whenever $y = x^2$. Therefore, all lines in this cylinder are parallel to the z-axis and the parabola is in the xy-plane.

Whichever variable is missing from the equation (for example, z is the "missing" variable in $y = x^2$), the cylinder is parallel to this missing variable's axis.

Whichever variables are present (for instance, in $y = x^2$ both x and y are present) denote which plane the cylinder resides.

Traces

One way we can visualize surfaces is to "flatten" them onto a two-dimensional plane for all three surfaces (xy, yz, and xz). This is useful for both cylinders and other surfaces.

The technical definition from Briggs Cochran is "the set of points at which the surface intersects a plane that is parallel to one of the coordinate planes. The traces in the coordinate planes are called the xy-trace, the yz-trace, and the xz-trace.

In layman's terms, you simply set whichever variable not mentioned in this trace and set it to 0. The line that results should be in $\mathbb{R}^2$ and is the trace for that plane.

Unless we're doing a trace of a circle, it's more than likely that the traces of each plane differ from each other.

12.2 Graphs and level curves

Two-variable functions

The key differentiating factor between $\mathbb{R}^2$ and $\mathbb{R}^3$ is that $\mathbb{R}^2$ has one independent variable and one dependent variable. On the other hand, $\mathbb{R}^3$ has two independent variables and one dependent variable. Functions, regardless of how many variables they have, always have one dependent variable.

Domain and range

Therefore, finding domain and range means that the domain is a relation in $\mathbb{R}^2$. We no longer specify the domain using simple inequalities in $\mathbb{R}^1$; we use a fancy notation that looks like $\{(x, y) : x > 5y\}$. The first part indicates which variables we have and the second part indicates for what relationship this domain holds true.

Level curves

Think of level curves as xy-traces but all parallel to each other and at different z values.

12.3 Limits and continuity in 3D

Limits

Limits in $\mathbb{R}^3$ are somewhat similar to limits in $\mathbb{R}^2$. The general premise holds, but some inherent differences lie in the fact that there are now two variables.

The general limit notation in $\mathbb{R}^3$ is:

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

Limit evaluation techniques from $\mathbb{R}^2$ mostly hold true for $\mathbb{R}^3$ as well. An exception is L'Hôpital's rule, which cannot be used when dealing with multiple variables unless you split the limit into two limits, each with just one variable.

3D limit rules

These rules from $\mathbb{R}^2$ still hold true in $\mathbb{R}^3$.

1. **Sum rule**:

$$\lim_{(x,y)\to(a,b)} \big(f(x, y) + g(x, y)\big) = \lim_{(x,y)\to(a,b)} f(x, y) + \lim_{(x,y)\to(a,b)} g(x, y)$$

2. **Difference rule**:

$$\lim_{(x,y)\to(a,b)} \big(f(x, y) - g(x, y)\big) = \lim_{(x,y)\to(a,b)} f(x, y) - \lim_{(x,y)\to(a,b)} g(x, y)$$

3. **Constant coefficient/constant multiple rule** (where λ is the constant):

$$\lim_{(x,y)\to(a,b)} (\lambda f(x,y)) = \lambda \lim_{(x,y)\to(a,b)} f(x,y)$$

4. **Product rule**:

$$\lim_{(x,y)\to(a,b)} \big(f(x,y) \cdot g(x,y)\big) = \lim_{(x,y)\to(a,b)} f(x,y) \cdot \lim_{(x,y)\to(a,b)} g(x,y)$$

5. **Quotient rule**:

$$\lim_{(x,y)\to(a,b)} \left(\frac{f(x,y)}{g(x,y)}\right) = \frac{\lim_{(x,y)\to(a,b)} f(x,y)}{\lim_{(x,y)\to(a,b)} g(x,y)}$$

6. **Exponent/power rule** (where n is the exponent/power):

$$\lim_{(x,y)\to(a,b)} (f(x,y)^n) = (\lim_{(x,y)\to(a,b)} f(x,y))^n$$

7. **Rational exponent/power rule** (where n is the numerator and m is the denominator of the power, $m \geq 0$, m is even, and n/m is reduced to lowest terms):

$$\lim_{(x,y)\to(a,b)} (f(x,y)^{\frac{m}{n}}) = (\lim_{(x,y)\to(a,b)} f(x,y))^{\frac{m}{n}}$$

Two-path test

In $\mathbb{R}^2$, we only had one independent variable. Therefore, it was "one dimensional" and limits were determined by approaching from the left and right.

In $\mathbb{R}^3$, we have two independent variables. Therefore, approaching a point can be done infinitely many ways.

In order to prove that a limit doesn't exist, we must prove that the approach is different between two different ways. Since there are infinitely many ways, this is when we get creative.

Continuity

The definition is the same as that of $\mathbb{R}^2$ except adapted for $\mathbb{R}^3$.

1. f is defined at (a,b);

2. $\lim_{(x,y)\to(a,b)} f(x,y)$ exists; and

3. $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$, or the value of the limit is equivalent to the actual function's value.

12.4 Partial derivatives

In $\mathbb{R}^2$, we evaluated derivatives with respect to the only independent variable available, x. In $\mathbb{R}^3$ however, since we have two independent variables, we must evaluate derivative with respect to each independent variable. These are called **partial derivatives**.

Why do we need them? Well, at a certain point, one independent variable's slope is different from another's. They change at different rates and therefore have their own derivatives. Therefore, we take partial derivatives for each independent variable.

The syntax behind partial derivatives can be expressed by either f_x or $\frac{\partial f}{\partial x}$.

To take a partial derivative, derive the selected variable (e.g. when we're given f_x or $\frac{\partial f}{\partial x}$ we derive x) and treat other variables as constants.

An example would be: $f(x, y) = 5x^2 + 3xy - 19y^2$. Then, $\frac{\partial f}{\partial x} = 10x + 3y - 0 = 10x + 3y$. Note that $\frac{\partial f}{\partial x} \neq \frac{\partial f}{\partial y}$, as $\frac{\partial f}{\partial y} = 3x - 38y$.

Partial derivatives are not to be confused with implicit derivatives. Review the section on implicit differentiation if you are unclear about it.

Partial derivatives continue to be valid for functions with three or more independent variables.

Higher-order partial derivatives

If you want to derive a certain variable twice or multiple times, then you would write something to the effect of $\frac{\partial^2 f}{\partial x^2}$ or f_{xx}.

If you want to derive both independent variables, the order in which you derive the variables does not matter. $\frac{\partial}{\partial y}(\frac{\partial f}{\partial x}(f(x, y))) = \frac{\partial}{\partial x}(\frac{\partial f}{\partial y}(f(x, y))) = f_{xy} = f_{yx}$.

Differentiability

A function is differentiable if $f_x(a, b)$ and $f_y(a, b)$ exist and $\Delta z = f(a+\Delta x, b+\Delta y) - f(a, b)$ equals

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

Implicit differentiation vs. partial differentiation

For a function $F(x, y)$, its implicit derivative can be found by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x}{F_y}$$

Note the implicit derivative is simply the opposite reciprocal of the partial derivatives divided by each other.

12.5 Chain rule

Applying the chain rule to $\mathbb{R}^3$ requires the use of partial derivatives. This means that chain rule will depend on how the variables are related to each other.

In $\mathbb{R}^2$, we saw that the chain rule was quite linear. For instance, suppose $w = f(u)$ and $u = g(t)$. Then,

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\mathrm{d}w}{\mathrm{d}u} \cdot \frac{\mathrm{d}w}{\mathrm{d}t}$$

We simply "unwrap" each layer at a time.

$$= f'(g(t)) \cdot g'(t)$$

The same principle applies in $\mathbb{R}^3$, but because there are multiple variables, there are multiple ways to reach our end goal. We must account for all of them by adding each partial derivative path up to form the sum total derivative. Remember, the top d variable is what we start with and the bottom is what we end with.

suppose $w = f(x, y)$ and $x = g(t)$ and $y = h(t)$. We start from w and end with t. Then,

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\partial w}{\partial x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial w}{\partial y} \cdot \frac{\mathrm{d}y}{\mathrm{d}t}$$

It can help to draw a tree diagram in this case. Put w at the top, then at the next layer, put x and y and link w to both of them. Then, link x to t (on the third layer) and y to t as well.

12.6 Directional derivatives and the gradient

So far, we've been finding derivatives along just the x and y dimensions. But what if we wanted to find the derivative in a specific direction?

We use a directional derivative. It is comprised of two components: a unit vector and a gradient.

A gradient the maximum rate of increase of a function, when considering any direction. For instance, if a 3D function represented a hill, then the gradient would point straight up to the top of the hill. It is a vector comprised of the partial derivatives at a certain point. It is denoted by $\nabla f(x, y)$.

The gradient is equal to:

$$\nabla f(a, b) = \langle f_x(a, b), f_y(a, b) \rangle$$

The notation for a directional derivative is:

$$D_{\vec{u}} \nabla f(x, y) = \nabla f(x, y) \cdot \vec{u}$$

$$= |\vec{u}||\nabla f(x, y)| \cos\theta = |\nabla f(x, y)| \cos\theta$$

Where $\vec{u}$ is the unit vector representing the direction in which we want to differentiate, $\nabla f(x, y)$ is the gradient, and θ is the angle between the directional derivative and the directional unit vector. ($|\vec{u}|$ always equals 1.)

The directional derivative is simply the dot product of these two vectors. This makes sense because the maximum rate of increase would be in the direction of the gradient and $\cos 0 = 1$. As the direction strays further away from the maximal increase direction, the directional derivative becomes closer and closer to zero. In fact, the direction orthogonal to the gradient is where there is zero change on the function. This again makes sense, as $\cos \frac{\pi}{2} = 0$.

To find the direction of maximum increase, find the gradient and divide it by its magnitude. For the maximal decrease, simply negate the vector.

Level curves and gradients

The level curve will always be orthogonal to the gradient. This is because the level curve's tangent vector points in a direction that the level curve doesn't change, whereas the gradient points in the direction of maximum change. Therefore, $\vec{t} \cdot \nabla f = 0$.

This is proved by implicit differentiation being equal to the opposite reciprocal of partial derivatives equation. How so? The slope of the tangent vector at (a, b) is the implicit derivative, which in vector form is $\langle f_y(a,b), -f_x(a,b)\rangle$. This is orthogonal to the gradient $\nabla f(a,b) = \langle f_x(a,b), f_y(a,b)\rangle$.

12.7 Tangent planes and linear approximations

A tangent plane is the analog to a tangent line in 2D. Tangent planes are for surfaces in 3D. To define tangent planes, you'll need to use partial derivatives. Similar to how we defined tangent lines in 2D, we'll do so in 3D as well, but for each dimension summed together.

To determine a tangent line, we needed the slope at that point and the points themselves. This drew on the linear equation's point-slope form. We'll do the same here.

There are two ways of describing a surface in $\mathbb{R}^3$:

1. Explicitly as a graph: $z = f(x, y)$
2. Implicitly as a level surface: $F(x, y, z) = C$

Finding tangent plane for implicit form surface

Consider a smooth curve $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ on surface $F(x, y, z) = c$. To find the tangent plane, we derive the function F using chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}t}[F(x(t), y(t), z(t))] = \frac{d}{dt}(c)$$

The derivative of a constant is zero. Per chain rule, we get:

$$\frac{\partial F}{\partial x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial F}{\partial y} \cdot \frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial F}{\partial z} \cdot \frac{\mathrm{d}z}{\mathrm{d}t} = 0$$

Therefore, we replace the derivative notation with partial derivatives and point-slope notations and we get the equation for a tangent plane:

$$F_x(a,b,c)(x-a) + F_y(a,b,c)(y-b) + F_z(a,b,c)(z-c) = 0$$

It turns out that $\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\rangle = \nabla F(x, y, z)$. Similarly, $\langle \frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t}\rangle = \vec{r}'(t)$. $\vec{r}'(t)$ is the tangent vector and is orthogonal to the gradient of function F. This means the normal vector of the tangent plane is parallel to the gradient. In fact, the gradient of F is the normal vector of the tangent plane.

Finding tangent plane for explicit form surface

Explicit form means a function written as $z = f(x,y)$. This is a special case of the implicit form, where $F(x,y,z) = f(x,y) - z = 0$. From this equation, we find that at the point $(a,b,f(a,b))$, $F_z(a,b,f(a,b)) = -1$. Therefore, the tangent plane formula is $f_x(a,b)(x-a) + f_y(a,b)(y-b) - (z - f(a,b)) = 0$. In a cleaner form:

$$z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

or

$$z = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$$

Linear approximation

The tangent plane is a good linear approximation for a function. Therefore, it is the exact same formula for finding tangent planes in explicit form, except $z = L(x,y)$.

$$L(x,y) = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$$

12.8 Extrema and optimization in 3D

Finding extrema in 3D is quite similar to 2D, except we have a slightly different second derivative test and we'll learn the power of Lagrange Multipliers in the next section.

In Calculus 1, we learned that **critical points** were where the derivatives were either equal to zero or did not exist. Through the second derivative tests, we can then hope to conclude whether this critical point were a minimum, maximum, or a saddle point.

In $\mathbb{R}^3$, we'll take the partial derivatives of the function and set them equal to zero, then find the critical points this way.

If we have a function f and $f_x(a,b) = f_y(a,b) = 0$ then a horizontal tangent plane exists. If either f_x or f_y does not exist, no tangent plane exists.

The second derivative test allows us to test our critical points. Let the discriminant $D = f_{xx} \cdot f_{yy} - (f_{xy})^2$. Then if $D(a,b) > 0$, you either have a local minimum or maximum. If $D(a,b) < 0$, there is a saddle point at (a,b). If $D(a,b) = 0$, the test is inconclusive.

How do we tell if the function is a local minimum or maximum? Take f_{xx} and see whether it is positive or negative. Beware of the counterintuitive answer. If $f_{xx} > 0$ (is positive), there is a minimum. If $f_{xx} < 0$ (is negative), there is a maximum.

3D optimization

In 3D, our domain is now two dimensional and defined by a relational equation. We should continue pursuing 3D optimization with this in mind.

1. Find your objective function (the one you will be optimizing) and the restriction function (the one that will be used to substitute variables into the objective function).

2. Choose one of the variables to be substituted by the restriction function into the objective function.

3. Set the domain. The domain is very important, as it serves as a restriction in which your function will be optimized. For instance, if this is a box's volume problem, you cannot have a length, width, or height of 0. That must be specified in the domain. The boundary area is where your domain is close to being false.

4. Take the first partial derivatives of your objective function.

5. Find the critical points by using systems of equations.

6. Use the second derivative test to determine the type of critical point it is.

7. See whether the local minima and/or maxima are absolute by checking where the boundary approaches. If the boundary isn't lower than the local minimum or higher than the local maximum, then that critical point can be said to be the absolute minimum or maximum.

12.9 Lagrange multipliers

Lagrange multipliers are useful tools to find extrema in 3D without second derivative tests. These are more efficient for optimization than the methods taught in the previous section.

In effect, given an objective function and a restriction function (defined in the previous section), let this objective function be $f(x, y)$ and the restriction function be $g(x, y)$ where they are both $= 0$. Then,

$$\nabla f(x, y) = \lambda \nabla g(x, y)$$

Where λ is a constant. ∇ indicates we take the gradient of $f(x, y)$ and $g(x, y)$.

Then, we set up a system of equations that include each term of $\nabla f(x, y)$ equal to $\lambda \nabla g(x, y)$ AND the original restriction function $g(x, y)$. Find lambda, then plug this value of lambda into the x equation. After you've found x, express the rest of the equations (except the restriction function itself) in terms of x and then plug those into the restriction function so that the restriction function is only in terms of x. (In lieu of x in the previous steps, you could solve for y instead if that's more efficient.)

Chapter 13

Multiple integration

Now that you have finished multivariable differential calculus, it is time to begin multivariable integral calculus with integration of multiple variables. There will be many parallels to partial derivatives.

13.1 Double integration over a rectangular region

In order to integrate in 3D, you'll use a double integral. One integral is used for each variable while the other variable is treated as a constant.

$$\iint_R f(x,y)\,\mathrm{d}A$$

In this case, R is the region over which you'll be integrating and A represents the area. The result would be a volume. To represent this by singular variables, we'll need to see the region R's bounds. Let's say $R = \{(x,y); 0 \le x \le 1, 2 \le y \le 3\}$. Then, we can write the integral of f as:

$$\int_0^1 \int_2^3 f(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \Big[\int_2^3 f(x,y)\,\mathrm{d}y\Big]\,\mathrm{d}x$$

As you can see, the y integral is inside and then the x integral is outside. We can actually reverse the order of this. The theorem that allows for this is called **Fubini's theorem**. For now, we can only say it holds true for double integration over a rectangular region (i.e. the limits of integration are constant).

$$\int_0^1 \int_2^3 f(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_2^3 \int_0^1 f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

13.2 Double integration over general regions

Whenever we need to integrate over a non-rectangular region, we'll integrate using a technique similar to the **area between the curves** technique that we used in Calculus 1. For double integrals, the region in which we integrate is going to be in 2D.

As opposed to the constant limits of integration in rectangular region double integration, in general regions the inner limits of integration is represented by an equation for the top and bottom bounds of the limits of integration.

This equation makes the double integral an **iterated integral**. Below is an example of an iterated integral:

$$\int_0^1 \int_{x^2}^x x + y \,\mathrm{d}y \,\mathrm{d}x$$

You can think of the limits of integration as functions of their respective limits of integration:

$$\int_{x=0}^{x=1} \int_{y=x^2}^{y=x} x + y \,\mathrm{d}y \,\mathrm{d}x$$

Let's solve this integral to get a better grasp of how we apply these limits of integration with variables in them.

$$= \int_{x=0}^{x=1} xy + \frac{y^2}{2}\Big|_{y=x}^{y=x^2} \mathrm{d}x$$

We now plug in x and x^2 into the integrand's y variables per the Second Fundamental Theorem of Calculus.

$$= \int_{x=0}^{x=1} x(x) + \frac{(x)^2}{2} - [x(x^2) + \frac{(x^2)^2}{2}] \,\mathrm{d}x$$

$$= \int_{x=0}^{x=1} x^2 + \frac{x^2}{2} - [x^3 + \frac{x^4}{2}] \,\mathrm{d}x = \int_{x=0}^{x=1} \frac{3x^2}{2} - x^3 - \frac{x^4}{2} \,\mathrm{d}x$$

From this point, you can solve the single-variable integral by yourself. The answer is $\frac{3}{20}$.

To flip these limits of integration, you will need to plot the region of integration and derive new limits of integration for both sets of limits of integration. Remember that the outer limit can be treated as vertical lines if the outer integral is with respect to x, or horizontal lines if the outer integral is with respect to y. You cannot use Fubini's theorem because the inner limits of integration are not finite values.

13.3 Change of variables for double integrals

In single-variable calculus, you used u**-substitution** and other similar techniques to change variables of integration to change to a form that is easier to integrate. In multivariable calculus, you will use a system of equations to create new mappings (one-to-one transformations, but don't worry about this term) of variables from (x, y) to the variables (u, v) which are arbitrarily related to x and y in some way.

The goal is to change the integral:

$$\iint_R f(x, y) \,\mathrm{d}A$$

to

$$\iint_S f(x(u, v), y(u, v)) \,\mathrm{d}A$$

Don't let the S or $f(x(u, v), y(u, v))$ panic you. $x(u, v)$ just means x will have some arbitrary relationship with u and v, such as $x = 2u + v$. S is the new region defined by u and v.

Also, in u-substitution, you may have gotten relationships by changing $\mathrm{d}x$ to $\mathrm{d}u$. For instance, sometimes you may have something like this: $\mathrm{d}u = 2\mathrm{d}x$. In this case, 2 is the derivative of the function x with respect to u. In multivariable calculus, you will need to do the same thing by taking the multivariable derivative:

$$\frac{\partial(u,v)}{\partial(x,y)}$$

This is called the **Jacobian determinant**. You add it to your newly transformed integral.

$$J(u,v) = \frac{\partial(u,v)}{\partial(x,y)} = \det \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

The equation for the newly transformed integral is:

$$\iint_R f(x,y)\,\mathrm{d}A = \iint_S f(g(u,v),h(u,v))\,|J(u,v)|\,\mathrm{d}A$$

Please note to include the **absolute value** of the Jacobian determinant! They mean it when they say $|J(u,v)|$.

13.4 Polar double integration

When the region of integration is circular, it is easier to integrate using polar coordinates instead of rectangular coordinates. In that case, we will change our variables from x and y to r and θ. We use the following relationships between the two sets of coordinate systems:

$$r^2 = x^2 + y^2$$
$$\theta = \arctan\left(\frac{y}{x}\right)$$

You will most likely be using the top transformation most often.

However, due to the symmetrical nature of circular limits of integration, it should be easy to spot. The outer limits of integration should be the radii of the circle and the inner limits of integration should be in the form $\sqrt{r^2 - x^2}$. This is assuming we integrate in the order $\mathrm{d}y\,\mathrm{d}x$.

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{x^2+y^2}\,\mathrm{d}y\,\mathrm{d}x$$

This integral fits this common pattern just described, so we can transform this integral into:

$$\int_0^{2\pi}\int_0^2 \sqrt{r^2}\cdot r\,\mathrm{d}r\,\mathrm{d}\theta$$

Notice that we applied the relationship $r^2 = x^2 + y^2$, substituting x and y with r. The extra r at the end is the Jacobian determinant for our polar integral! Note that changing from the rectangular coordinates (x,y) to (r,θ) does require you to use the Jacobian determinant. Thankfully, it is the same for all polar integration. $J(r,\theta) = r$.

We get this after everything is simplified:

$$\int_0^{2\pi}\int_0^2 r^2\,\mathrm{d}r\,\mathrm{d}\theta$$

From here, the problem is trivial.

Note that if the original integral's limits of integration were changed, it may halve or quarter the original region. You must make up for this change in the limit of integration that is with respect to θ. For instance, if you are given:

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \sqrt{x^2+y^2}\,\mathrm{d}y\,\mathrm{d}x$$

That is one-fourths of the area of the original integral's region of integration. Therefore the transformed integral should be:

$$\int_0^{\pi/2} \int_0^2 r^2\,\mathrm{d}r\,\mathrm{d}\theta$$

Notice that 2π is now $\pi/2$.

13.5 Triple integration

Triple integration involves integrating on a third variable. In the rectangular coordinate system, this variable is z. Let's use an example from our previous section to segue into triple integrals.

$$\int_0^{2\pi} \int_0^R h\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{2\pi} \int_0^R \int_0^h 1\,\mathrm{d}r\,\mathrm{d}\theta = V_{\text{cylinder}} = \pi r^2 h$$

Here, we can change the integrand h into the limit of integration from 0 to h. Note that these are integrals for the volume of a cylinder. In this cylinder volume case, h is held constant, so we can move it into the third integral's limits of integration. However, if h weren't constant, then we would have to stick with a double integral.

Now, a triple integral is useful when we integrate for all three coordinate systems. Currently, we are putting two of the three coordinate systems in the limits of integration, and the third is represented by the integrand. However, if we put all three coordinates into the limits of integration, the integrand can represent the change of the inside composition of the solid region of integration, such as density in physics.

Practice problems

These below practice problems vary in difficulty, but should cover the important kinds of problems you need to be able to solve in order to succeed in a calculus course. Some problems are (and should be) trivial, like $\sin \frac{\pi}{12}$, but other problems will be very rigorous, such as the later surface area problems.

0.1 Precalculus review problems

Problem 0.1.1 *Calculate* $\sin \frac{\pi}{12}$. *(Kallman)*

Problem 0.1.2 *Calculate* $\sin \frac{3\pi}{8}$. *(Kallman)*

Problem 0.1.3 *Calculate* $\sin^2(\frac{3\pi}{2})$ *without squaring.*

Problem 0.1.4 *Simplify* $\csc^3(x) \cdot (\frac{\sec^2(x)-1}{\sec^2(x))})$.

Problem 0.1.5 *Simplify* $\frac{(\sec(x)+\csc(x))^2 - \frac{4}{\sin^2(2x)}}{2\csc^2(x)}$.

Problem 0.1.6 *If* $\cos^2(z) = 0.3$, *what is* $\sin^2(z)$*?*

0.2 Limits problems

Problem 0.2.1 *Calculate* $\lim_{x\to 3} \frac{x^2-9}{x-3}$

Problem 0.2.2 *Calculate* $\lim_{x\to -2^+} \frac{x-2}{x^2-4}$

Problem 0.2.3 *Calculate* $\lim_{x\to +\infty} \frac{x^2+5}{2x^2-9}$

Problem 0.2.4 *Calculate* $\lim_{x\to 0} \frac{\sin x}{x}$.

Problem 0.2.5 *Calculate* $\lim_{x\to 0} \frac{\sin(3x)\tan(7x)}{x\tan(11x)}$. *(Kallman)*

Problem 0.2.6 *Give an* (ε, δ) *proof that* $\lim_{x\to 2} x + 2 = 4$.

Problem 0.2.7 *Give an* (ε, δ) *proof that* $\lim_{x\to 3} x^3 = 8$. *(Kallman)*

0.3 Differentiation problems

Problem 0.3.1 *Let* $f(x) = \sin(x)$. *Calculate* $(D^{999} f)(x)$. *(Kallman)*

Problem 0.3.2 *Let* $f(x) = 12x^4 + 2x^5$. *Calculate* $(D^5 f)(x)$. *(Kallman)*

Problem 0.3.3 *Let* $f(x) = \frac{x^{13}-1+\cos^{19}(x)-\cos^{19}(1)}{x-1}$. *Calculate* $\lim_{x\to 1} f(x)$. *(Kallman)*

Problem 0.3.4 *Let* $f(x) = \cos\big(\sin\big(\frac{x}{1-x}\big)\big)$. *Find* $f'(x)$. *(Kallman)*

Problem 0.3.5 *Let* $f(x) = \frac{2x}{4+x^2}$. *What are the zeroes of* $f'(x)$*? (Kallman)*

Problem 0.3.6 *Let* $f(x) = \sin\big(\cos\big(\sqrt{x^5 - 19x^2 + 53x + 13}\big)\big)$. *Compute* $f'(x)$. *(Kallman)*

Problem 0.3.7 *Let* $f(x) = \sec^{23}(x)\tan^{31}(x)$. *Compute* $f'(x)$.

0.4 Applications of differentiation problems

Problem 0.4.1 *Use implicit differentiation to find the second derivative (D^2y) of the hyperbola defined by the equation $b^2x^2 - a^2y^2 = a^2b^2$. (Kallman)*

Problem 0.4.2 *Find the critical points and the local extrema of the function $f(x) = x^4 - 4x^3$ on $\mathbb{R}$. (Kallman)*

Problem 0.4.3 *Find the critical points and the local extrema of the function $f(x) = \frac{x^3}{x+1}$. (Kallman)*

Problem 0.4.4 *Find the inflection points and determine the concavity of $f(x) = 3x^4 + 6x^2 - 12x$.*

Problem 0.4.5 *Calculate $\lim_{x\to 0} \frac{x-\sin(x)}{x^3}$. (Kallman)*

Problem 0.4.6 *Find the equation for the tangent line to the curve $\tan(xy) = x$ at the point $(1, \frac{\pi}{4})$. (Kallman)*

Problem 0.4.7 *For the equation $y^3 - x^2 = 4$, use implicit differentiation to express y'' in terms of x and y. (Kallman)*

Problem 0.4.8 *Calculate $\lim_{x\to \frac{\pi}{4}} \frac{\sec^2(x)-2\tan(x)}{1+\cos(4x)}$.*

Problem 0.4.9 *Calculate $\lim_{x\to 1} \frac{\sqrt{x}-\sqrt[4]{x}}{x-1}$.*

Problem 0.4.10 *Scott has been given 60 ft of fencing on his rectangular property. Timber Creek borders one side of his property, which he does not need to fence. He wants to fence the most area possible with this amount of fencing. What will the length and width of this fenced area be?*

Problem 0.4.11 *Judge Judy needs your help. A defendant of hers, Louise, has just lost a lawsuit. She must now garnish 3.5% of her pay for one year and do x amount of community service days per year. Assuming she gets paid $\$400$ every day, optimize x so she can do the most community service per day while still being able to earn $\$120,000$ this year before federal and state income taxes. Help Judge Judy determine the best amount of community service days Louise should do while still being able to earn a living.*

Problem 0.4.12 *Compute $\lim_{x\to 0^+} \sin(x)\sqrt{\frac{1-x}{x}}$. (Kallman)*

Problem 0.4.13 *A right triangle has lengths of length h and r and a hypotenuse of length 4. It is revolved around the leg of length h to sweep out a right circular cone. What values of h and r maximize the volume of the cone? (Kallman)*

Problem 0.4.14 *Prove that $|\sin^2(x) - \sin^2(y)| \leq 2|x-y|$ for all $x, y \in \mathbb{R}$. Hint: use the Mean Value Theorem. (Kallman)*

Problem 0.4.15 *Find the equations for the lines tangent and normal to the curve $x^2 + xy + 2y^2 = 28$ at the point $(-2, -3)$. (Kallman)*

Problem 0.4.16 *Compute $\lim_{x\to 0} \frac{2\cos(x)-2+x^2}{x^4}$. (Kallman)*

Problem 0.4.17 *Find the critical points, local extreme values, convexity, points of inflection and asymptotes of $f(x) = \frac{1}{x-1} - \frac{1}{x-2}$. (Kallman)*

0.5 Integration problems

Problem 0.5.1 *Let* $F'(x) = 20$*. What is* $F(x)$*?*

Problem 0.5.2 *Let* $F'(x) = \cos(x)$*. What is* $F(x)$*?*

Problem 0.5.3 *Let* $F'(x) = 3\sec^2(x)$*. What is* $F(x)$*?*

Problem 0.5.4 *Calculate* $\int 2x^7 + 3x^6 + 9x + 7\,\mathrm{d}x$*.*

Problem 0.5.5 *Calculate* $\int 2x(x^2+3)^3\,\mathrm{d}x$*.*

Problem 0.5.6 *Calculate* $\int \sqrt{x - x\sqrt{x}}\,\mathrm{d}x$*. (Kallman)*

Problem 0.5.7 *Calculate* $\int \frac{\sec^2(x)}{\sqrt{1+\tan(x)}}\,\mathrm{d}x$*. (Kallman)*

Problem 0.5.8 *Calculate* $\int x^2\tan(x^3+\pi)\sec^2(x^3+\pi)\,\mathrm{d}x$*. (Kallman)*

Problem 0.5.9 *Find* $H'(3)$ *given that* $H(x) = \frac{1}{x}\int\limits_3^x [3t + 2H'(t)]\,\mathrm{d}t$*. (Kallman)*

Problem 0.5.10 *Calculate* $\int\limits_1^{\sqrt{2}} x^3(x^2-1)^7\,\mathrm{d}x$*. (Kallman)*

Problem 0.5.11 *Compute* $\int_0^1 \frac{x+3}{\sqrt{x+1}}\,\mathrm{d}x$*. (Kallman)*

Problem 0.5.12 *Compute* $\int \frac{1}{\sqrt{x\sqrt{x}+x}} dx$*. (Kallman)*

0.6 Applications of integration problems

Problem 0.6.1 *Region R is bounded by* $y = \frac{7}{8}x$*,* $x = 0$*,* $x = 6$*, and the x-axis. What is the volume of the solid of revolution made from the region R revolved about the x-axis?*

Problem 0.6.2 *Region R is bounded by* $y = \frac{\sin(x)}{x}$*,* $y = \frac{\cos(x)}{x}$*,* $x = \frac{\pi}{4}$*, and* $x = \frac{\pi}{2}$*. What is the volume of the solid of revolution made from the region R revolved about the y-axis? (Kallman)*

Problem 0.6.3 *Let R be the region in the first quadrant bounded by the coordinate axes and the curve* $y = 1 - x^2$*. A solid has base R and the cross section through the solid, perpendicular to the base and parallel to the y-axis, are squares. Find the volume of the solid. (Kallman)*

Problem 0.6.4 *Find the length of the graph of* $f(x) = (x+3)^{3/2}$ *from* $x = 0$ *to* $x = 44$*. (Kallman)*

Problem 0.6.5 *Find the volume of the solid generated by revolving the region between* $f(x) = x^2$ *and* $g(x) = 2x$ *about the y-axis. (Kallman)*

Problem 0.6.6 *Find the volume of the solid generated by revolving the region between* $f(x) = x^2$ *and* $g(x) = 2x$ *about the x-axis. (Kallman)*

Problem 0.6.7 *Find the area of the surface generated by revolving the graph of* $f(x) = \frac{1}{3}x^3$ *on* $[0, 2]$ *about the x-axis. (Kallman)*

Problem 0.6.8 *Let R be the region in the first quadrant bounded by the coordinate axes and the curve* $y = 1 - \cos(x)$ *on* $[0, \pi]$*. A solid has base R and the cross section through the solid that's perpendicular to the base and parallel to the y-axis are equilateral triangles. Find the volume of the solid. (Kallman)*

Problem 0.6.9 *Find the length of the graph of* $f(x) = \frac{1}{6}x^3 + \frac{1}{2}x^{-1}$ *from* $x = 1$ *to* $x = 3$*. (Kallman)*

Problem 0.6.10 *Find the area of the surface generated by revolving the graph of* $f(x) = 2\sqrt{1-x}$ *on* $[-1, 0]$ *about the x-axis. (Kallman)*

Problem 0.6.11 *Let* Ω *be the region bounded by the curve* $y = x^{2/3} + 1$*, bounded to the left by the y-axis, and bounded above by the line* $y = 5$*. Find the volume of the solid generated by revolving* Ω *about the y-axes. (Kallman)*

Problem 0.6.12 *Find the volume enclosed by the surface obtained by revolving the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *about the x-axis. (Kallman)*

Problem 0.6.13 *A base of a solid is the region enclosed by the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$*. Find the volume of the solid given that each cross section perpendicular to the x-axis is an isosceles triangle with base in the region and altitude equal to one-half the base. (Kallman)*

0.7 Late transcendentals problems

Problem 0.7.1 *For a certain function,* $f(2) = 3$ *and* $f'(2) = 8$*. What is* $(f^{-1})'(2)$*?*

Problem 0.7.2 *Calculate* $\frac{d}{dx}\ln(12x)$*.*

Problem 0.7.3 *Calculate* $\frac{d}{dx}\ln(7e)$*.*

Problem 0.7.4 *Calculate* $\int \frac{12x^2}{4x^3+7}\,\mathrm{d}x$*.*

Problem 0.7.5 *Calculate* $\int \frac{\sin(2x)}{\sin^2(x)}\,\mathrm{d}x$*.*

Problem 0.7.6 *Calculate* $\int e^{\sin x}\cos x\,\mathrm{d}x$*.*

Problem 0.7.7 *Calculate* $\frac{d}{dx}(x^{\ln x})$*.*

0.8 Integration techniques problems

Problem 0.8.1 *Calculate* $\int \frac{\mathrm{d}x}{1+e^x}$*. (Kallman)*

Problem 0.8.2 *Calculate* $\int \frac{\mathrm{d}x}{1+\ln^2(x)}$*. (Kallman)*

Problem 0.8.3 *Calculate* $\int \frac{x\,\mathrm{d}x}{x^2+2x+5}$*. (Kallman)*

Problem 0.8.4 *Calculate* $\int x^2\, 2^x \,\mathrm{d}x$. *(Kallman)*

Problem 0.8.5 *Calculate* $\int \frac{x}{\sqrt{4x^2-9}} \,\mathrm{d}x$.

Problem 0.8.6 *Calculate* $\int \frac{x^2}{\sqrt{4x^2-9}} \,\mathrm{d}x$.

Problem 0.8.7 *Calculate* $\int \frac{x^2}{\sqrt{x^2+4}} \,\mathrm{d}x$.

Problem 0.8.8 *Calculate* $\int \frac{x^2}{\sqrt{9-x^2}} \,\mathrm{d}x$. *(Kallman)*

Problem 0.8.9 *Calculate* $\int \frac{\mathrm{d}x}{x(x^2+1)}$.

Problem 0.8.10 *Calculate* $\int \frac{\mathrm{d}x}{x(x^2+1)^2}$. *(Kallman)*

Problem 0.8.11 *Calculate* $\int \frac{\mathrm{d}x}{x(x+2)^2}$.

Problem 0.8.12 *Calculate* $\int \frac{\mathrm{d}x}{x(x+1)(x+2)(x+3)}$.

Problem 0.8.13 *Show that* $\int_1^{+\infty} \frac{3\,\mathrm{d}x}{e^x+5}$ *converges. (Kallman)*

Problem 0.8.14 *Calculate* $\int_e^{+\infty} \frac{\mathrm{d}x}{x \ln x}$. *(Kallman)*

Problem 0.8.15 *Calculate* $\int_{100}^{+\infty} \frac{\ln x}{x} \,\mathrm{d}x$.

0.9 Sequences and series problems

Problem 0.9.1 *Evaluate this geometric series if it exists or state that it diverges.* $\sum_{k=3}^{+\infty} \frac{3 \cdot 4^k}{7^k}$ *(Briggs Cochran)*

Problem 0.9.2 *Evaluate this geometric series if it exists or state that it diverges.* $\frac{1}{16} + \frac{3}{64} + \frac{9}{256} + \frac{27}{1024} + \cdots$

Problem 0.9.3 *Evaluate this telescoping series if it exists or state that it diverges.* $\sum_{k=1}^{+\infty} (\arctan(k+1) - \arctan(k))$

Problem 0.9.4 *Calculate* $\sum_{\ell=2}^{+\infty} \frac{3(-4)^{\ell-1} + 5(-6)^{\ell+1}}{11^\ell}$. *(Kallman)*

Problem 0.9.5 *Use the integral test to evaluate this series.* $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$

Problem 0.9.6 *Use the ratio test to determine whether this series converges.* $\sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!}$

Problem 0.9.7 *Use the root test to determine whether this series converges.* $\sum_{k=1}^{\infty} \left(\frac{k}{k+1}\right)^{2k^2}$

Problem 0.9.8 *Prove the following corollary.* ***Corollary*** $\forall\, n \in \mathbb{N}, \sum_{k=1}^{\infty} \frac{(k!)^n}{(nk)!} = \frac{1}{n^n}$ *per the Ratio Test.*

Problem 0.9.9 *Use the alternating series test to determine whether this series converges or diverges.* $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k^{10}+2k^5+1}{k(k^{10}+1)}$

Problem 0.9.10 *Pick a test and use it to determine whether this series converges.* $\frac{1}{2^2} + \frac{2}{3^2} + \frac{3}{4^2} + \cdots$

0.10 Power series problems

Problem 0.10.1 *Expand* $p(x) = 4x^3 - 3x^2 + 5x - 1$ *in terms of* $x - 1$ *(i.e.* $a = 1$*).*

Problem 0.10.2 *Find the Taylor expansion for* $f(x) = x^3 + 5x^2 + 2x + 7$ *about* $a = 1$.

Problem 0.10.3 *Write out Taylor's Formula with Remainder for* $f(x) = \ln x, a = 2, n = 3$.

Problem 0.10.4 *Find the* n*th-order Taylor polynomials for the following functions centered at the given point* a*, for* $n = 0, 1,$ *and* 2. $f(x) = \arctan(x) + x^2 + 1, a = 1$

Problem 0.10.5 *Find the first four terms of the Taylor expansion of* $\tan x$ *about* $a = 0$.

0.11 Calculus 3 problems

No problems are provided for Calculus 3.

0.12 Humorous problems

$$\frac{1}{n}\sin x = ?$$
$$\frac{1}{\not n} si\not n x = ?$$
$$six = 6$$

$$\int \frac{1}{\text{cabin}}\,\mathrm{d}\,\text{cabin} = \log_e(\text{cabin}) + C$$

Or, more accurately,

$$\int_1^{\text{cabin}} \frac{1}{x}\,\mathrm{d}x = \log_e(\text{cabin})$$

End matter

Dear reader,
Thank you for reading NoBS Calculus. I hope it has made your long journey through the world of calculus easier.
–Jeffrey Wang